Mathematical relationships with tree size distributions

Lauri Mehtätalo

University of Joensuu

Meeting in Norway, March 2007

Outline of the presentation

Let X be a variable characterizing the size of a tree in a forest stand.
The most common variable used for tree size is tree diameter. Examples of other alternatives are tree height, crown diameter, crown area, basal area, and tree volume.
The within stand (or within plot) variability in tree size is accounted for through tree size distribution. Tree (unweighted) size distribution of a stand is defined as the probability that a randomly selected tree from the target stand is smaller than a fixed size x,

$$F(x) = P(X \leq x)$$

The corresponding density is

$$f(x) = F'(x)$$

An alternative interpretation for diameter distribution is the proportion of trees with the size smaller than x.

In the following, I will explain how transformations and weighting can be used to compute different stand characteristics from a diameter distribution. These relationships can also be utilized in formulating prediction and recovery models for diameter distributions.
As efficient procedures for evaluating integrals and solving numerical equations are nowadays freely available, closed form solutions for the desired functions are not needed. If we are able to write the formulas as integrals and systems of equations, numerical methods can be used to solve the equations and evaluate the integrals numerically.

The relationships between different tree variables can be accounted for through transformations. Let Y be another variable for tree size, which is obtained from X through a monotonic transformation $g(X)$. The distribution of Y is

$$F_Y(y) = F_X(g^{-1}(y)) \quad \text{if } g \text{ is increasing}$$
$$F_Y(y) = 1 - F_X(g^{-1}(y)) \quad \text{if } g \text{ is decreasing}$$

Example 1: The distribution of height

Let D be tree diameter and H tree height. The diameter distribution is two-parameter Weibull

$$F_D(d) = 1 - \exp\left[-\left(\frac{d}{\beta}\right)^{\alpha}\right] .$$

and height-diameter curve is

$$h = 1.3 + a \exp\left(\frac{b}{d}\right) , \tag{1}$$

where parameters have known values $\alpha = 4$, $\beta = 15$, $a = 25$ and $b = -5$.
Solving (1) for d gives $g^{-1}(h) = \frac{b}{\ln\left(\frac{h-1.3}{a}\right)}$.

The distribution of tree height becomes

$$F_H(h) = 1 - exp\left[-\left(\frac{b}{\beta \ln\left(\frac{h-1.3}{a}\right)}\right)^{\alpha}\right] ,$$

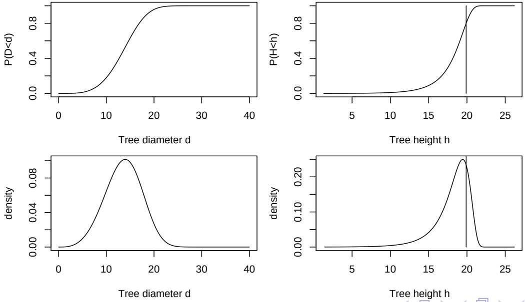
which is no more a Weibull distribution. The density is

$$f_H(h) = \frac{\alpha}{\beta}\left(\frac{b}{\beta \ln\left(\frac{h-1.3}{a}\right)}\right)^{\alpha-1} \exp\left[-\left(\frac{b}{\beta \ln\left(\frac{h-1.3}{a}\right)}\right)^{\alpha}\right] \frac{-b}{(h-1.3)\left[\ln\left(\frac{h-1.3}{a}\right)\right]^2}$$

Example 1: The distribution of height (continued)

The mean height is $\bar{H} = \int_0^{H_{max}} u f_H(u) \mathrm{d}u = 18.12103$ m.
The height distribution of dominant trees is obtained by truncating the height distribution and rescaling it to unity. Dominant height is the expected value of that distribution (we assume N=500 stems/ha). The limit of dominant trees is $F_H^{-1}((N-100)/N) = 19.89572$ m and we get:

$$H_{dom} = \frac{N}{100} \int_{F_H^{-1}((N-100)/N)}^{H_{max}} u f_H(u) \mathrm{d}u = 20.44752$$



In standard diameter distributions, frequencies are proportional to the number of stems. It is often more convenient to have the frequencies proportional on some other characteristics, most commonly on basal area. The density of a basal area weighted diameter distribution is

$$f_D^G(x) = \frac{d^2 f_D^N(d)}{\int_0^\infty u^2 f_D^N(u)\mathrm{d}u} \qquad (2)$$

The unweighted density is obtained from the basal area weighted diameter distribution, f_D^G, through

$$f_D^N(x) = \frac{d^{-2} f_D^G(d)}{\int_0^\infty u^{-2} f_D^G(u)\mathrm{d}u}\,. \qquad (3)$$

In general the density of a weighted diameter distribution is

$$f_D^w(d) = \frac{w(d) f_D(d)}{\int_0^\infty w(u) f_D(u)\mathrm{d}u}\,.$$

The nominator scales teh density to unity. It is just the integral of the numerator over the range of d. i.e., the mean weight $w(d)$, e.g., mean basal area, height or volume.

Weighted distributions can be used for fitting assumed distributions to a sample obtained through weighted sampling. Examples of such situations are

- Angle-count sampling is used and a specific distribution is asumed for unweighted distribution (Van Deusen, Gove and Patil).
- Fixed-area plots are used and a specific distribution is asumed for basal area weighted distribution.
- Sample plot radius varies according to tree diameter.
- Data are censored, for example, small trees have not been measured or cannot be observed from air
- Observation probability depends on tree size according to a known function

Example 2: Overlapping crowns in a Poisson stand

In an aerial forest inventory, crown radius Z describes tree size. It is asumed to follow a two-parameter Weibull distribution. We assume that a tree remains unobserved if the tip is within a crown of a larger tree. Otherwise, the tree is observed and crown area is determined correctly. It can be shown that the probability for a tree being observed depends on crown radius according to

$$w(z) = \pi^{\frac{1}{\int_0^\infty t^2 f_Z(t|\alpha,\beta)dt} \int_z^\infty t^2 f_Z(t|\alpha,\beta)dt},$$

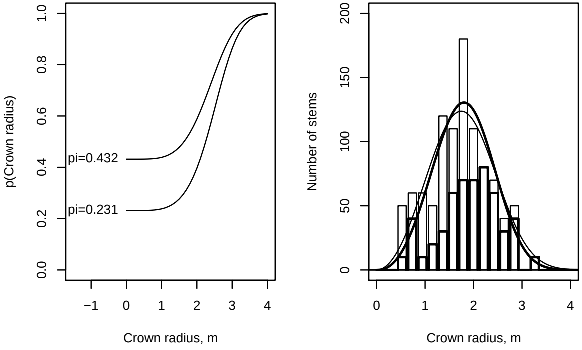
where π is the expected canopy closure, replaced with its observed value in applications.
The distribution of observed crown areas is

$$f_Z^w(z|\alpha,\beta) = \frac{w(z|\alpha,\beta) f_Z(z|\alpha,\beta)}{\int_0^\infty w(u|\alpha,\beta) f_Z(u|\alpha,\beta) \mathrm{d}u}.$$

Parameters α and β can be estimated by fitting the above distribution to the observed sample of crown radii. Stand density can then be estimated as $\lambda = -\frac{\ln(\pi)}{E(Z)}$.
The same principle can be applied with a discrete distribution, too, by weighting each observation by inverse of $w(z)$.

Example 2: Overlapping crowns in a Poisson stand (continued)

Figure: The plot on the left shows the probability of a tree being observed. The plot on the right shows an example of an estimated distribution using simulated data. The Histogram with thick lines is the observed sample, the histogram with thin lines shows all trees of the plot. the thin line shows the true underlying Weibull distribution and the Thick line the estimated distribution.



The distribution of diameter dependent variables, can be approximated by classifying diameter and applying the desired transformation to the mean tree of the class. The total is computed by summing up the class-specific values.

A more convenient way is to define a weighted distribution and integrate it over the desired range of size. Let $t(x)$ define a transformation of size. The following relationship holds

$$Tf_X^T(x) = Nf_X^N(x)t(x)\,, \tag{4}$$

where T is the total of t. The total of t between specified sizes is $N\int_{x_1}^{x_2} f_X^N(u)t(u)\mathrm{d}u$. For example, if tree size is diameter and $t(d) = \pi d^2/4$, the total basal area is $G = N\int_0^\infty f_D^N(u)t(u)\mathrm{d}u$.

If we start with a weighted distribution, we can first got to unweighted distribution and then to the desired weight. For example, we can go from a basal area weighted distribution to a volume weighted distribution through

$$Vf_X^V(d) = Nf_D^N(d)v(d) = G\frac{f_D^G(d)}{\pi d^2/4}v(d)$$

The total volume is $V = \frac{40000G}{\pi}\int_0^\infty f_D^G(u)u^{-2}v(u)\mathrm{d}u$.

Example 3: Another way to compute mean and dominant height

Let g be the height model we specified before,

$$h(d) = 1.3 + a \exp\left(\frac{b}{d}\right)$$

with $a = 25$ and $b = -5$ and assume the unweighted diameter distribution to be Weibull with $\alpha = 4$ and $\beta = 15$.
Another way to compute the mean height is to compute the sum of tree heights and divide it by N. N cancels and we get

$$\bar{H} = \int_0^\infty f_D^N(u)h(u)\mathrm{d}u = 18.12104$$

The same principle with dominant height yields

$$H_{dom} = \frac{N}{100} \int_{F_D^{-1}((N-100)/N)}^\infty f_D^N(u)h(u)\mathrm{d}u = 20.44752\,,$$

where the diameter limit for dominant trees is $F_D^{-1}((N-100)/N) = 16.89506$ cm.

Example 4: Effect of a harvest on total volume

Half of stems will be harvested from below from a stand where F_D^G is Weibull with $\alpha = 4$ and $\beta = 15$ and $G = 20m^2/ha$. Heights come from the Korf curve with $a = 25$ and $b = -5$. The volume function is

$$v(d,h) = 0.022927d^{1.915}0.99146dh^{2.825}(h-1.3)^{-1.535}$$

The unweighted diameter distribution fulfils

$$Nf_D^N(d) = 40000G/\pi d^{-2} f_D^G(d)$$

The total number of stems is $N = \frac{40000G}{\pi}\int_0^\infty d^{-2} f_D^G(u)\mathrm{d}u = 2006.007$. Solving $f_D^N(d) = 0.50N$ for d we see that the diameter limit for harvest is $d = 10.36$. The distribution of volume is $\dot{f}_D^V(d) := Vf_D^V(d) = 40000G/\pi f_D^G(d)d^{-2}v(d,h(d))$. The remaining volume and basal area are

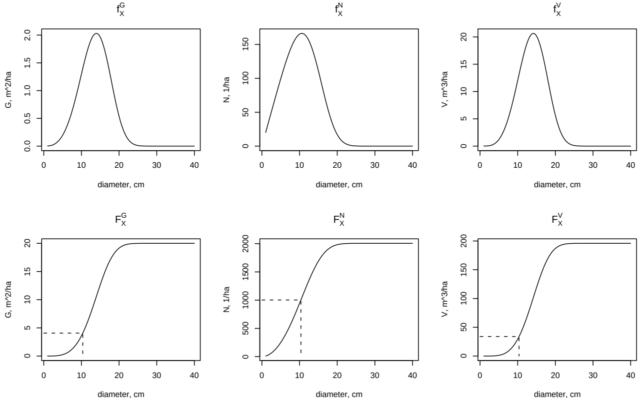
$$V_{remaining} = \int_{10.36}^\infty \dot{f}_D^V(u)(d)u = 161.9796m^3/ha$$

and

$$G_{remaining} = G\int_{10.36}^\infty f_D^G(u)(d)u = 15.93095m^2/ha\,.$$

Example 4 (continued): Effect of a harvest on total volume

Figure: Diameter distributions weighted by basal area, number of stems and total volume. The dashed lines demonstrate the effect of harvesting 50% of the stems from below on total volume and basal area.



Example 5. PRM using volume, mena height and basal area

Assume that F_D^G is a two-parameter Weibull distribution. What are the values for α and β in a stand with V=200 m^3/ha, G=20m^2/ha and $\bar{H}$=16 m. The volume function and H-D curve are the same as in Example 4.
The parameter values should fulfil

$$\begin{aligned} \frac{40000G}{\pi}\int_0^\infty f_D^G(u|\alpha,\beta)u^{-2}v(u,h(u))\mathrm{d}u - V &= 0 \\ \frac{\frac{40000G}{\pi}\int_0^\infty f_D^G(u|\alpha,\beta)u^{-2}h(u)\mathrm{d}u}{\frac{40000G}{\pi}\int_0^\infty f_D^G(u|\alpha,\beta)u^{-2}\mathrm{d}u} - \bar{H} &= 0 \end{aligned}$$

The solution to this nonlinear system of equations is $\alpha = 3.800971$ and $\beta = 17.072494$. The solution was found using a Newton-Raphson method for a nonlinear system of equations with numerical differentiation.

Example 6. PPM and PRM models for Weibull parameters

(A work with Jukka Nyblom) We assumed that G, N, DGM and mean height are known. The known relationships between stand variables were utilized to specify how the predictors enter into the PPM model.
When $f_D^G(d)$ is Weibull density, the following equations hold:

$$\beta = \frac{DGM}{\ln 2^{1/\alpha}}$$
$$\frac{\pi}{4\Gamma(1-2/\alpha)(\ln 2)^{2/\alpha}} = \frac{G}{N \cdot DGM^2},$$

where $\Gamma()$ is the gamma function.
PRM: the above system was solved for α and β.
PPM: the following models were fitted to pine data using 2SLS

$$shtr_i = a_\alpha + b_\alpha \cdot xtr1_i[+c_\alpha \bar{H}_i] + e_{\alpha i}$$
$$\beta_i = a_\beta + b_\beta \cdot xtr2_i[+c_\beta \bar{H}_i] + e_{\beta i},$$

where $shtr_i = \frac{\pi}{4\Gamma(1-2/\alpha_i)(\ln 2)^{2/\alpha_i}}$, $xtr1_i = \frac{G_i}{N_i \cdot DGM_i^2}$ and $xtr2_i = \frac{DGM_i}{(\ln 2)^{1/\widehat{\alpha}_i}}$.

Example 6 (continued). PPM and PRM models for Weibull parameters

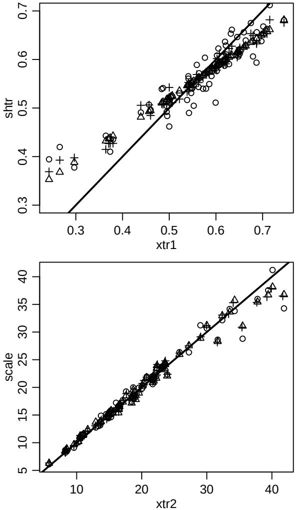
Table: The parameter estimates of PPM1 and their standard errors

	Estimate	s. e.	*p*-value
Model for shtr (σ^2=0.0283, R^2=0.862)			
constant	0.194	0.0186	1.84e-15
$xtr1$	0.653	0.0326	8.9e-16
Model for scale, (σ^2=0.885, R^2=0.988)			
constant	0.783	0.293	0.00951
$xtr2$	-1.566	0.263	1.23e-7
DGM	2.744	0.289	9.56e-14

Table: The parameter estimates of PPM2 and their standard errors

	Estimate	s. e.	*p*-value
Model for shtr (σ^2=0.0265, R^2=0.883)			
constant	0.169	0.0208	2.43e-11
$xtr1$	0.695	0.0331	$<$2e-16
DGM	0.00401	0.00121	0.00151
H	-0.00502	0.00166	0.00373
Model for scale, (σ^2=0.8321, R^2 0.989)			
constant	0.546	0.290	0.0645
$xtr2$	-1.133	0.266	7.14e-5
DGM	2.134	0.306	2.40e-9
H	0.195	0.0522	0.0004

Example 6 (continued). PPM and PRM models for Weibull parameters



Observations (circles), PPM1 predictions (triangles) and PPM2 predictions (crosses) of the models for transformed shape (upper) and untransformed scale (lower). The reference line corresponds to the theoretical values of parameters, $a = 0$ and $b = 1$.

Example 6 (continued). PPM and PRM models for Weibull parameters

Table: Results from comparisons between different estimation methods in the modeling and test datasets.

		ML fit	PPM1	PPM2	PRM	Partial recovery
				Modeling data		
Volume	RMSE	1.19	1.29	1.11	1.22	1.22
	Bias	-0.039	-0.053	0.033	-0.418	-0.416
Error index	Mean	6.10	6.43	6.30	6.56	6.29
				Test data		
Volume	RMSE	0.296	0.952	0.688	0.652	1.05
	Bias	-0.013	-0.522	-0.292	-0.013	-0.84
Error index	Mean	6.64	7.68	7.62	8.33	7.59

Example 6 (continued). PPM and PRM models for Weibull parameters

True and predicted weighted distributions of three selected stands of the test data.
Histogram: true distribution
Dashed: ML estimate
Thin solid: PPM2
Dotted: PRM

Volume

	True	ML-fit	PPM2	PRM
A	36.5	36.5	36.6	36.7
B	81.1	81.3	81.7	81.1
C	102.9	103	103.7	102.9

Error index

	ML-fit	PPM2	PRM
A	1.0	1.2	1.2
B	3.6	4.7	6.2
C	4.8	7.0	13.5