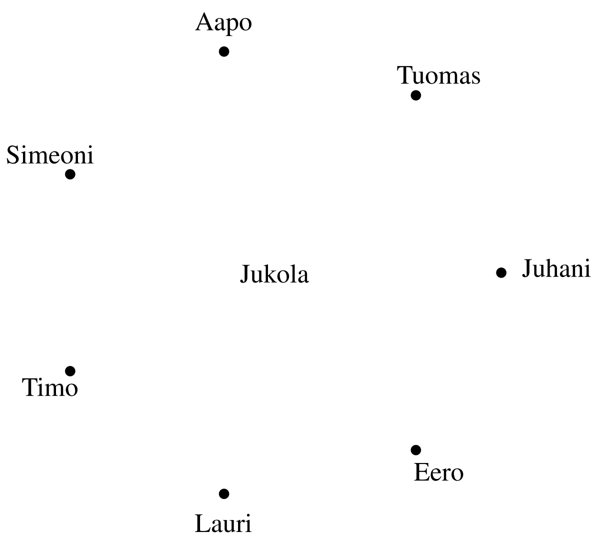
Multidimensional Geometry

Eric Lehman

February 2018


Aapo
Tuomas
Simeoni
Jukola
Juhani
Timo
Eero
Lauri

Chapitre 1

The five Platonic solids

In this chapter we are following Coxeter. The regular convex polyhedra (or polyhedrons) are the simplest solid figures, if we except the sphere.

§ 1. Pyramids, prisms and antiprisms

A *convex polygon* such as the regular polygon with n edges or sides and n vertices is a finite region of the plane intersection of half-planes. Analogously, a convex polyhedron is a finite region of space enclosed by a finite number of planes. The part of each plane that is cut off by the other planes is a polygon called *face*. Any common side of two faces is an *edge*.

Right regular prisms

We shall be concerned solely with "right regular" pyramids and prisms. The faces of such pyramids are one base which is a regular polygon and side faces which are congruent isosceles triangles. The faces of a right regular prism are two regular n-gons connected by n rectangles. Question : given a vertex, how are the faces having this vertex in common ?

The height of such a prism can always be adjusted so that the rectangles become squares. What do we get when $n = 4$?

The cube is *regular*, because all the faces are congruent regular polygons, all the edges are congruent and at all vertices we have the same configuration of faces.

Right regular pyramids

The height of n-gonal pyramid can be adjusted so that the isosceles triangles become equilateral if $n < 6$. A triangular pyramid is a *regular tetrahedron,* regular since the four faces are congruent.

Antiprisms

Let us rotate one of the bases of a right regular prism on an n-gone by an angle $\frac{1}{2}\frac{2\pi}{n}$ and connect the two bases by $2n$ isosceles triangles. Choose the height such that all these triangles become equilateral. We get a uniform polyhedron with 3 triangles and 1 n-gone at each vertex.

When $n = 3$ the antiprism is an *octahedron*.

When $n = 5$ we can combine it with two pentagonal pyramids, one on each base, to get the regular *icosahedron*.

The dodecahedron

Draw on a cardboard a regular pentagon F and then 5 pentagons each one having one side in common with F. Cut it out and fold it to get a bowl. Build a second bowl the same way and then put them together.

Schläfli symbols

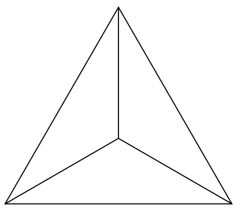
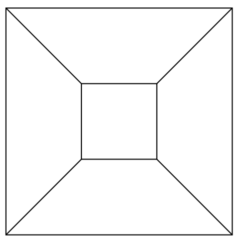
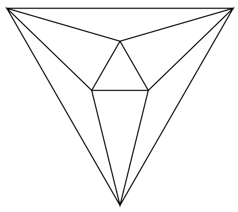
Each polyhedron is characterized by a Schäfli symbol $\{p, q\}$ which means the faces are p-gones and that there are q faces at each vertex.

We put $V =$ number of vertices, $E =$ number of edges and $F =$ number of faces.

§ 2. Drawings and models

Schlegel diagram

Construct a polyhedron using sticks for the edges and draw the image in perspective from a point of view just outside the center of one face near enough to get an image of the edges of that face which surrounds all the other edges. The result is a Schlegel diagram of the polyhedron. Here are the results for the tetrahedron $\{3, 3\}$, the cube $\{4, 3\}$ and the octahedron $\{3, 4\}$.

Draw the Sclegel diagram of the dodecahedron $\{5, 3\}$ and the icosahedron $\{3, 5\}$:

§ 3. Euler's formula

Any polyhedron that can be represented by a Schlegel diagram is said to be *simply connected* or *Eulerian*, because its numerical properties satisfy Euler's formula

$$V - E + F = 2$$

This formula is valid for any connected graph formed by a finite number of points joined by edges decomposing the plane in non overlapping regions.

Proof. Build up the graph beginning with one single point by adding a new edge at each step. The new edge has to start from an existing point and has to end either at an existing point or at a new point. In the first case you have to add 1 to E and 1 to F since the edge has cut a face in two. In the second case F is constant but V and E increase both by 1. In both cases $V - E + F$ is unchanged at each step and we have begun with $V = 1$, $E = 0$ and $F = 1$. □

V, E and F as functions of p and q

In a regular polyhedron we have

$$qV = 2E = pF$$

since each edge has 2 endpoints and each edge is a side for 2 faces.

Now remember the very convenient formula

$$\text{if} \quad \frac{A_1}{B_1} = \frac{A_2}{B_2} = \cdots = \frac{A_N}{B_N} \quad \text{then} \quad \frac{A_1}{B_1} = \frac{A_2}{B_2} = \cdots = \frac{A_N}{B_N} = \frac{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_N A_N}{\lambda_1 B_1 + \lambda_2 B_2 + \cdots + \lambda_N B_N}$$

Thus

$$\frac{V}{\frac{1}{q}} = \frac{E}{\frac{1}{2}} = \frac{F}{\frac{1}{p}} = \frac{V - E + F}{\frac{1}{q} - \frac{1}{2} + \frac{1}{p}} = \frac{2}{\frac{2p - pq + 2q}{2pq}} = \frac{4pq}{2p + 2q - pq}$$

from what we get

$$V = \frac{4p}{2p + 2q - pq} \quad , \quad E = \frac{2pq}{2p + 2q - pq} \quad \text{and} \quad F = \frac{4q}{2p + 2q - pq}$$

Since these quantities have to be positive, the numbers p and q have to satisfy $2p + 2q - pq > 0$, or equivalently

$$(p - 2)(q - 2) < 4$$

Put $p' = p - 2$ and $q' = q - 2$, the positive integers p' and q' have to be such that their product is less or equal to 3. The possibilities are

$$(1,1) \quad \text{or} \quad (2,1) \quad \text{or} \quad (1,2) \quad \text{or} \quad (3,1) \quad \text{or} \quad (1,3)$$

The five possible regular polyhedra are thus such that $\{p,q\}$ is one of these

$$(3,3) \quad \text{or} \quad (4,3) \quad \text{or} \quad (3,4) \quad \text{or} \quad (5,3) \quad \text{or} \quad (3,5)$$

But these five polyhedra exist !

§ 4. The three spheres associated to a regular polyhedron

Given a regular polyhedron, let us call O the center of it. How would you define such a center and does it exist ?

You may define three spheres associated to that regular polyhedron :

— the *circumsphere* which passes through all vertices

— the *midsphere* which is tangent to all the edges (the contact points are the midpoints of the edges)

— the *insphere* which is tangent to each face (the contact points are the center of the faces)

We skip the computation of the radii of these spheres to another year !

§ 5. Reciprocal polyhedra

The reciprocal platonic solid to $\{p,q\}$ is $\{q,p\}$: just take the polyhedron you get by choosing as vertices the centers of the faces.

Exercises

Exercise 1. Draw a cube $ABCDEFGH$. Select four points A, C, F and H such that no two points belong to a common edge. Describe the polyhedron $ACFH$.

Exercise 2. Let T be a regular tetrahedron. How can you construct a cube whose set of vertices contains all the vertices of T ? Is this cube unique ?

Exercise 3. Let T be a regular tetrahedron. Describe the polyhedron whose vertices are the midpoints of the edges of T. You may use the result of exercise 2.

Exercise 4. Give a description of a regular octahedron as a two-pyramid.

Exercise 5. Describe a solid having 5 vertices and 6 equilateral triangular faces. Why is it not a "regular pentahedron" ?

Exercise 7. Describe the section of a regular tetrahedron by the plane midway between two opposite edges.

Exercise 8. Describe the section of a cube by the plane midway between two opposite vertices.

Exercise 9. Describe the section of a regular dodecahedron by the plane midway between two opposite planes.

From now on, we suppose we take the coordinates relatively to a frame $(0, \vec{\imath}, \vec{\jmath}, \vec{k})$ where the basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$ is orthonormal.

Exercise 10. Let C_1 be the cube of edge 1 with all vertices with nonnegative coordinates, one vertex at the origin and the edges parallel to the three axes of the frame. Show that the coordinates of the vertices of C_1 are (X, Y, Z) where $X \in \{0, 1\}$, $Y \in \{0, 1\}$ and $Z \in \{0, 1\}$.

Let C_2 be the cube of edge 2 with its center at the origin and the edges parallel to the three axes of the frame. Show that the coordinates of the vertices of C_2 are $(\pm 1, \pm 1, \pm 1)$.

Where is the center of dilatation that relates C_1 and C_2 ?

Exercise 11. Describe the solid defined by

$$|x| + |y| + |z| \leqslant 1$$

Exercise 12. Let τ be the golden ratio : $\tau = \frac{\sqrt{5}+1}{2}$. A golden rectangle is a rectangle where the length is equal to the breadth multiplied by τ. Let us look at the three following golden rectangles :

— one in the yOz plane with vertices $(0, \pm\tau, \pm 1)$;

— one in the zOx plane with vertices $(\pm 1, 0, \pm\tau)$;

— one in the xOy plane with vertices $(\pm\tau, \pm 1, 0)$;

Show that these 12 points are the 12 vertices of a regular icosahedron. Draw a picture showing the three rectangles.

Chapitre 2

Four-dimensional regular polytopes

Four-dimentional geometry is the most difficult geometry in the matter of polytopes. From dimension five and up to infinity it is simpler ! So don't get afraid by any four-dimentional hypercube or other "24-cell".

§ 1. Drawings

1.1 Draw a 4-dimensional simplex

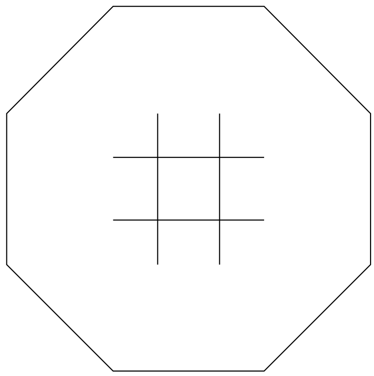
It the generalization of the regular tetrahedron. Think of a pyramid with a vertex in the new dimension and having a regular tetrahedron as base.

Here again we count the number of vertices : , the number of edges : , the number of faces : , the number of cells : .

This polytope is called a *4-dimensional simplex*.

1.2 Draw a 4-dimensional cube

It the generalization of the cube. We'l get a *hypercube* or *tesseract* (or "8-cell" or "measure polytope").



Moving pictures : https ://funnyjunk.com/Hypercube/hdgifs/6112424/

https ://en.wikipedia.org/wiki/Tesseract

Here again : number of vertices : , number of edges : , number of faces : , number of cells : .

§ 2. Schläfli symbols

A regular polytope is characterized by its Schläfli symbol $\{p, q, r\}$. We have the symbol $\{p, q\}$ for the cells (all congruent, of course).

Question : why do we say "congruent" instead of "equal" ?

The faces of the cells are such that each face $\{p\}$ belongs to 2 cells. But the edges are belonging to several cells : r cells. We can look at r in an other way : Take the midpoints of the edges coming out from one vertex : they make a regular polyhedron $\{q, r\}$. Thus $\{p, q, r\}$ is a contraction of $\{p, q\}$ and $\{q, r\}$.

4-simplex : $\{3, 3, 3\}$ (V,E,F,C)=(, , ,)

Hypercube : $\{4, 3, 3\}$ (V,E,F,C)=(, , ,)

Orthoplex : $\{3, 3, 4\}$ (V,E,F,C)=(, , ,)

24-cell : $\{3, 4, 3\}$ (V,E,F,C)=(24,96,96,24)

120-cell : $\{3, 3, 5\}$ (V,E,F,C)=(600,1200,720,120)

600-cell : $\{3, 3, 5\}$ (V,E,F,C)=(120,720,1200,600)

https ://en.wikipedia.org/wiki/120-cell

https ://en.wikipedia.org/wiki/600-cell

Exercises

Exercise 1. Let H be the hypercube of edge 2 with its center at the origin and the edges parallel to the four axes of the frame $(O,\vec{\imath},\vec{\jmath},\vec{k},\vec{\ell})$. Find the coordinates of the vertices of H.

What are the coordinates of the centers of the cells of the hypercube H?

Exercise 2. Find the coordinates of an orthoplex having its vertices on the axes of the orthonormal frame $(O,\vec{\imath},\vec{\jmath},\vec{k},\vec{\ell})$.

Exercise 3. Guess what is the analogue of Euler's theorem for polytopes in 4-dimensional geometry?

Chapitre 3

Basic notions

Everything begins with Euclid (about 300 before JC). The nickname "Euclis" means "the good key". He builds geometry entirely upon common notions and axioms. Now it is easy to say that he missed one, the Pash axiom : "If a line intersects one side of a triangle, it intersects at least one of the two others. Remark that in this context the sides of a triangle ABC are the segments BC, CA and AB.

The non-euclidean geometries were invented/discovered in the 19th century. We shall go back to this later on if we have time.

Finally Hilbert put an endpoint to that history with his book "Grundlagen der Geometrie" in 1899 : there are points, lines *and* planes *(these words have no mathematical meaning, but you may keep the intuitive meaning you have inherited from Euclid). The axioms relating these objects are classified in 5 groups : axioms of incidence, axioms of order, axioms of congruence, axioms of continuity and axioms of parallelism. In this chapter we will be concerned mostly by the axioms of incidence and a little by those of parallelism.*

Nowdays it is usual to make use of a model constructed on the set theory which is equivalent to the axiomatic formulation of Geometry by Hilbert. Starting with set theory and the recursivity axiom one constructs the set of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and then $\mathbb{R}^3$. We can put a structure of linear space on $\mathbb{R}^3$. What we have got looks very much like 3D-geometry, but it is not yet satisfactory. Two main defects are : 1°) no distance is defined : we do not have any Pythagoras theorem ; 2°) one of the points in $\mathbb{R}^3$ is a very special point : the origin or null vector. We'll take care of Pythagoras in the following chapters. For the time being, just forget that the the origin is special and think that any vector $\overrightarrow{\omega}$ may be the origin The expressions "the point A lies on the line ℓ" or "ℓ goes through A" mean simply $A \in \ell$. Similarly "the point A lies on the plane h" or "h goes through A" mean $A \in h$. But notice that "the line ℓ lies in the plane h" means $\ell \subset h$ we use the symbol $\subset$ as a synonym of $\subseteq$.

§ 1. The axioms of incidence

Let us denote by $\mathcal{E}$ the set of points, by $\mathcal{L}$ the set of lines and by $\mathcal{P}$ the set of planes.

1.1 Relations between points and lines

1. There are at least two distinct points. : $\operatorname{card}\mathcal{E} \geqslant 2$;

2. There is one and only one line that contains two distinct points. Given two distinct points A and B the unique line ℓ that contains them will be denoted AB

$$\forall A \in \mathcal{E}\ \forall B \in \mathcal{E} \quad A \neq B \Longrightarrow \exists! \ell \in \mathcal{L}\ A \in \ell \text{ and } B \in \ell$$

3. Every line contains at least two distinct points : $\forall \ell \in \mathcal{L}\ \operatorname{card}\ell \geqslant 2$.

1.2 Relations between points and planes

4. There are three points that do not all lie on the same line.

$$\exists (A, B, C) \in \mathcal{E}^3 \ \forall \ell \in \mathcal{L} \quad \{A, B, C\} \not\subset \ell$$

5. For any three points that do not lie on the same line there is one and only one plane that contains them.

$$\forall (A, B, C) \in \mathcal{E}^3 \left(\forall \ell \in \mathcal{L} \quad \{A, B, C\} \not\subset \ell \implies \exists! P \in \mathcal{P} \quad \{A, B, C\} \subset P \right)$$

6. Any plane contains at least three points : $\forall P \in \mathcal{P} \quad \text{card}\, P \geqslant 3$.

1.3 Relations between points and planes

7. If a line lies on a plane then every point contained in the line lies on that plane.

$$\forall \ell \in \mathcal{L} \ \forall P \in \mathcal{P} \quad \ell \subset P \implies \left(\forall A \in \mathcal{E} \ \ A \in \ell \implies A \in P \right)$$

8. If a line contains two points which lie on a plane then the line lies on the plane.

$$\forall P \in \mathcal{P} \ \forall (A, B) \in \mathcal{E}^2 \left(A \neq B \ \textbf{and} \ \{A, B\} \subset P \right) \implies \left(\forall \ell \in \mathcal{L} \quad \{A, B\} \subset \ell \implies \ell \subset P \right)$$

1.4 Dimension of space

9. If two planes both contain a point then they also contain a line.

$$\forall P \in \mathcal{P} \ \forall P' \in \mathcal{P} \qquad P \cap P' \neq \varnothing \implies \exists \ell \in \mathcal{L} \quad \ell \subset P \cap P'$$

10. There are at least four points that do not all lie on the same plane.

$$\exists (A, B, C, D) \in \mathcal{E}^4 \ \forall P \in \mathcal{P} \quad \{A, B, C, D\} \not\subset P$$

Remark. The first four axioms (which do not refer to planes) are the plane geometry axioms, while the remaining are the space axioms. Out of the various Theorems that can be proved we note

Theorem 1. Given a line and a point not on it there is one and only one plane that contains the line and the point.

Theorem 2. Given a pair of lines which meet in a point there is one and only one plane that contains the lines.

Theorem 3. Given four points that do not all lie on a plane, there is no line containing three of these points.

§ 2. Desargues theorem

The most general statement of Desargues Théorem is the following :

Theorem. Two triangles are in perspective axially if and only if they are in perspective centrally.

Vocabulary. Let ABC and $A'B'C'$ be two triangles. If there is a point S and three lines p, q and r *concurrent* in S (all three lines go through the point S) such that A and A' belong to p, B and B' belong to q, C and C' belong to p, then the triangles are said to be *centrally perspective* or *in perspective centrally*.

Let the lines $a = BC$, $b = CA$ and $c = AB$ be the sides of the triangle ABC and similarly $a' = B'C'$, $b' = C'A'$ and $c' = A'B'$ be the sides of the triangle $A'B'C'$. If there is a line s containing the points $P = a \cap a'$, $Q = b \cap b'$ and $R = c \cap b'$, then the triangles are said to be *axially perspective*or *in perspective axially*.

We keep the notations ABC, $A'B'C'$, a, b, c, a', b', c' throuh the whole paragraphe. The triangles ABC and $A'B'C'$ are supposed to be real triangles, that is to say that the vertices are not collinear.

2.1 Generic statement in space

Quasi theorem. Let ABC and $A'B'C'$ be two triangles in space such that the planes Π and Π' of these triangles intersect along a line s and such that $A' \neq A$, $B' \neq B$, $C' \neq C$. We suppose that a and a' are not parallel, that b and b' are not parallel and that c and c' are not parallel. If the lines AA', BB' and CC' are concurrent in a point S, then the lines a and a' intersect each other in a point P, $b \cap b' = Q$ and $c \cap c' = R$ all three belonging to the line s.

Proof. The lines AA' and BB' go through the point S. Since $B \neq B'$, we have $S \neq B$ or $S \neq B'$. By a change of names we may suppose that $S \neq B$. Let us call H the plane containing the line AA' and the point B. The points of the line AA' belong to H, thus $S \in H$. The points S and B belong to H thus the line SB lies on H. Since there is only one line through B and B' and since this line contains S distinct from B, the lines SB and BB' are equal (that is the same line). Since this line is in H, we have $B' \in H$. Thus the four points A, A', B and B' are coplanar (all 4 points belong to the plane H). The lines $c = AB$ and $c' = A'B'$ are coplanar and not parallel, thus they have a common point R.

Since $R \in c \subset \Pi$ and $R \in c' \subset \Pi'$, we have $R \in \Pi \cap \Pi' = s$. Similarly, we get $P \in s$ and $Q \in s$. The three points belong to a common line, the line s. □

2.2 Generic statement in the plane

Quasi theorem. Let ABC and $A'B'C'$ be two triangles in a plane Π such that $A' \neq A$, $B' \neq B$, $C' \neq C$. We suppose that a and a' are not parallel, that b and b' are not parallel and that c and c' are not parallel. Let us put $P = a \cap a'$, $Q = b \cap b'$ and $R = c \cap c'$, If the lines AA', BB' and CC' are concurrent in a point S, then the points P, Q and R are collinear.

Proof. Let's choose a point Σ which does not belong to Π and an other point Σ' on the line ΣS distinct from S and from Σ. The lines ΣA and $\Sigma A'$ are in a plane : the plane determined by the lines SAA' and $S\Sigma\Sigma'$. If the lines ΣA and $\Sigma' A'$ are parallel, just change by a small amount the point Σ' and they become secant. Lat us call α the point $\Sigma A \cap \Sigma' A'$ and similarly $\beta = \Sigma B \cap \Sigma' B'$ and $\gamma = \Sigma C \cap \Sigma' C'$. Let us call π the plane through α, β and γ and s the line which is common to the planes π and Π. The line $\beta\gamma$ is in the plane ΣBC. Let us call P_1 the intersection of the lines $\beta\gamma$ and ΣBC. We define in the same way $P_1' = \beta\gamma \cap \Sigma' B'C'$. Now P_1 and P_1' belong th the line $\beta\gamma$ and the plane Π, thus $P_1 = P_1'$ and so this is a point common to BC and to $B'C'$. Then $P_1 = P_1' = P$. Thus $P \in s$. Similarly $Q \in s$ and $R \in s$. □

Comment. We have not taken into account parallelism. Two solutions : reason in affine geometry and discuss all the different cases or reason in projective geometry where there are no parallels : two parallel lines are lines having in common one "new" point : the point at infinity which is the direction or the set of all the lines parallel to the two we are considering.

2.3 In the affine plane

cf. https ://mathcs.clarku.edu/ djoyce/java/round/desargues.html

Weak form of Desargues theorem. Let p, q and r be three lines concurrent in a point S or parallel. Let ABC and $A'B'C'$ be two triangles without any common vertex and such that A and A' belong to p, B and B' belong to q and C and C' belong to r. If $BC \| B'C'$ and $CA \| C'A'$, then $AB \| A'B'$.

Strong form of Desargues theorem. Given two triangles ABC and $A'B'C'$ and three distinct lines p, q and r such that $A \neq A'$, $B \neq B'$, $C \neq C'$, A and A' belong to p, B and B' belong to q and C and C' belong to r. Then the three lines $p = AA'$, $q = BB'$ and $r = CC'$ are parallel or concurrent if and only if one of the following three conditions is valid :

• the lines BC and $B'C'$ intersect, CA and $C'A'$ intersect and also AB and $A'B'$ intersect and the points $P = BC \cap B'C'$, $Q = CA \cap C'A'$ and $R = AB \cap A'B'$ are collinear ;

• two among the three couples of lines BC et $B'C'$, AC et $A'C'$, AB et $A'B'$ are couples of intersecting lines and the last couple contains two parallel lines which are also parallel to the line joining the former intersection points ; for instance $P = BC \cap B'C'$, $Q = CA \cap C'A'$ and the lines AB and $A'B'$ are parallel and parallel to the line PQ ;

• all three couples of lines are couples of parallel lines : $BC \parallel B'C'$, $CA \parallel C'A'$ and $AB \parallel A'B'$.

2.4 In the projectif plane

Theorem. Let ABC and $A'B'C'$ be two non flat triangles (the vertices are not collinear) such that $A \neq A'$, $B \neq B'$ and $C \neq C'$ and let p, q and r be three distinct lines such that $p = BC \cap B'C'$, $q = CA \cap C'A'$ and $r = AB \cap A'B'$:

if the lines p, q and r are intersecting in one point S, then the points $P = a \cap a'$, $Q = b \cap b'$ and $R = c \cap c'$ belong to a common line s ;

if the three points $P = a \cap a'$, $Q = b \cap b'$ and $R = c \cap c'$ belong to a common line, then the three lines $p{=}AA'$, $q{=}BB'$ and $r{=}CC'$ are intersecting in a common point S.

Exercises

Exercise 1. Using the axioms show the theorem 1 : Given a line and a point not on it there is one and only one plane that contains the line and the point.

Exercise 2. Using the axioms show the theorem 2 : Given a pair of lines which meet in a point there is one and only one plane that contains the lines.

Exercise 3. Using the axioms show the theorem 3 :Given four points that do not all lie on a plane, there is no line containing three of these points.

Exercise 4. Among the axioms of incidence, which one is not necessary ?

Exercise 5. Show that parallelism is an equivalence relation in the set $\mathcal{L}_2$ of all the lines lying in a common plane.

Show that parallelism is an equivalence relation in the set $\mathcal{P}$ of all the planes in 3D-space.

Show that parallelism is an equivalence relation in the set $\mathcal{L}_3$ of all the lines in 3D-space.

What is the name of the elements of the quotients $\mathcal{L}_2/\parallel$ and $\mathcal{L}_3/\parallel$?

Exercise 6. On a plane draw three circles C, D and E with distinct radii and outside of each other. Draw the common tangents to D and E which let D and E be in the same half-plane (called outer tangents), and call P the intersection point of these two tangents. Define similarly the point Q, intersection of the outer tangents to E and C and define similarly the point R, intersection of the outer tangents to C and D.

1°) Guess a theorem.

2°) Give a proof of your theorem using 3D-geometry (you do not need to make any new drawing, just look at the one you have in another way ! and you do not need any words when you "see" your solution).

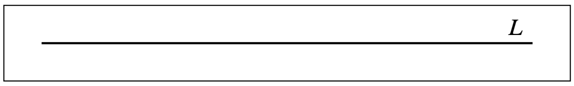
3°) Give another proof of your theorem.

Exercise 7. Let A and B be two distinct points in an n-dimensional space. Let $c = AB$ be the line through A and B. Similarly let A' and B' be two distinct points in the same n-dimensional space and such that $A \neq A'$, $B \neq B'$ and $c \neq c'$. Show that the lines AA' and BB' are concurrent if and only if c and c' are concurrent.

Exercise 8. Is it possible to generalize Desargues's theorem to 4 dimensions ?

Chapitre 4

One-dimensional geometries

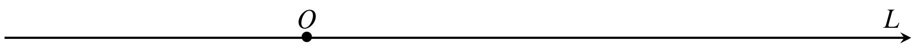


The line above is a set of points. If we take one point O nothing happens. If we take two points, O and I, we can just say that $O = I$ or $O \neq I$. Now if we take a third point M, we can compare the length and orientation from O to M relative to the length and orientation from O to I. In fact we describe the line, given two distinct points O and I, by the set of real numbers $\mathbb{R}$. The distance from O to I will be our unit. In physics we may take the cm. In mathematics we do not care, any two distinct points are enough. In geometry it is sometimes convenient to think that one choice has been made and sometimes it is better to leave that question open.

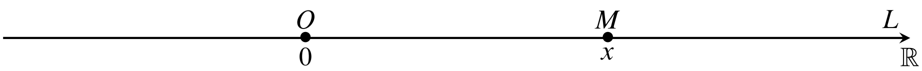
§ 1. Oriented 1-dimensional euclidean geometry

1.1 Description of a physical line

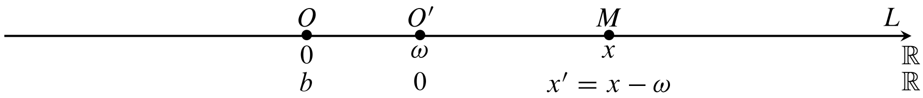
Let us denote by L the set of points of the line. We can orient the line by putting an arrow at one end (of course, we draw only a finite part of the line supposed to go on indefinitely in both ends). We choose one point O.



Now to each point M we can associate a real number x in a one to one map : you just measure the length OM in cm and put $+$ sign if M is on the right hand side of O, a $-$ sign if M is on the left hand side of O and put $x = 0$ if $M = O$. Let us call that map by c as "coordinate". Thus there is a bijection of L onto $\mathbb{R}$: $c : L \to \mathbb{R}, M \mapsto c(M) = x$.



What happens if we choose another point O' ? We get another bijection. Let us call it c' and put $x' = c'(M)$. Can we compute x' knowing x ? Of course we must say where we took the point O', that is we have to suppose that we know $c(O') = \omega$. Then $c'(M) = c(M) - \omega$, or $x' = x - \omega$.



The formula giving x'in term of x is the explicit form of the map $c' \circ c^{-1} : \mathbb{R} \to \mathbb{R}$. Let us put $b = -\omega$. We can summarize our study in the following way : we have a set L and a family of bijections $\mathcal{B}$ of L onto $\mathbb{R}$ such that if c and c' belong to $\mathcal{B}$, then there is a real number b such that for all x in $\mathbb{R}$

$$(c' \circ c^{-1})(x) = x + b$$

1.2 Vectors

We are working in one dimension. Thus things are in a way too simple. Here we are using the fact that given a field F, F is one-dimensional vector space on itself (vector space means the same thing as linear space, but since we want to talk about vectors, we'll use the term "vector space").

If we take two points A and B, we notice that the real number $c(B) - c(A)$ do not depend on the choice of the bijection c. We say that this number is intrinsic and $\mathbb{R}$ being a vector space on itself, we may call it "vector" and use the notation $\overrightarrow{AB}$. If C and D are two other points on the line we can compute $\overrightarrow{CD}$. Thus $\overrightarrow{AB} = \overrightarrow{CD}$ means $c(B) - c(A) = c(D) - c(C)$, and this relation is valid independently of the choice of c in $\mathcal{B}$.

1.3 Mathematical definitions

Now forget (nearly) everything we have done and forget also about the physics. We start from scratch.

Definition. Let F be a field. An *oriented Euclidean line over the field* F is a couple $(L, \mathcal{B})$, where L is a set and $\mathcal{B}$ is a set of bijections of L onto F such for any c and c' in $\mathcal{B}$ there is an element b in F such that

$$c' \circ c^{-1} : F \to F, x \mapsto x + b$$

We say that the change of coordinates is given by

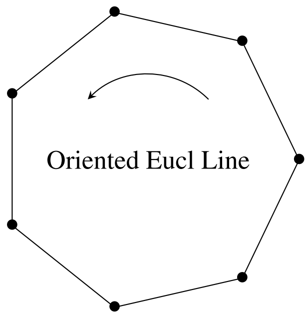
$$x' = x + b$$

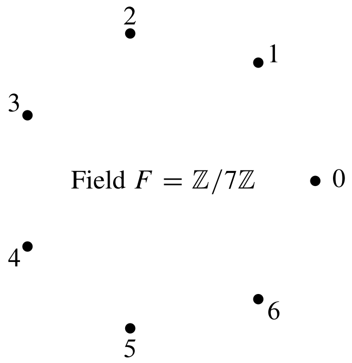
Remark. Usually the adjective "Euclidean" is reserved for geometry on the real field $\mathbb{R}$. But I do not know any generic denomination for a general field F.

Example 1. Let L be the set of electric potential. We suppose we know what is higher potentials or what is lower potentials and we know the unit of difference of potentials. That explains that you never can say that the electric potential is this or that but only that the difference of electric potential is equal to so and so much, measured in Volts.

Example 2. Any line which is not necessary a straight line, but such that one may measure distances along that line and without multiple points and such that it is directed like a one-way road.

Example 3. Let us take the field $F = \mathbb{Z}/7\mathbb{Z}$. We may sketch the line in the following way





Proposition and Definition. Let $(L, \mathcal{B})$ be an oriented Euclidean line. Given two points A and B in L, $c(B) - c(A)$ is an element of F independent of the choice of the bijection c in $\mathcal{B}$. It is called a *vector* and denoted $\overrightarrow{AB}$. The set of vectors is a vector space called the *vector space associated to the line* L. Thus

$$\overrightarrow{AB} = c(B) - c(A)$$

Proposition. The map $L \times L \to F$, $(A, B) \mapsto \overrightarrow{AB}$ has the following properties :

— for every point A, the map $L \to F$, $B \mapsto \overrightarrow{AB}$ is a bijection ;

— for all points A, B and C in L, we have $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Notation. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F. Let $\overrightarrow{u}$ be an element of F,we denote by $M + \overrightarrow{u}$ the point N such that $\overrightarrow{MN} = \overrightarrow{u}$. In that case, we use also the notations $\overrightarrow{u} = N - M$.

Proposition and Definition. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F and let $\overrightarrow{u}$ be a vector in the vector space associated with L.

The map $T_{\overrightarrow{u}} : L \to L, M \mapsto M + \overrightarrow{u}$ is a bijection called *translation of vector* $\overrightarrow{u}$.

It is easy to check that the set of translations of $(L, \mathcal{B})$ is a group isomorphic to the group $(F, +)$ through

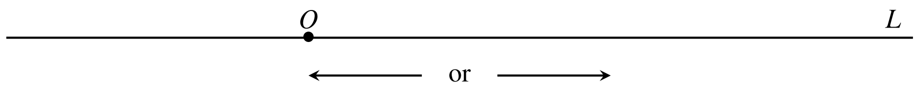
$$T_{\overrightarrow{u}} \circ T_{\overrightarrow{v}} = T_{\overrightarrow{u}+\overrightarrow{v}}$$

§ 2. 1-dimensional Euclidean geometry

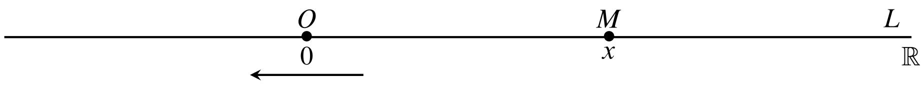
Now we skip orientation. We'll follow the same track as for the oriented 1-dimensional euclidean geometry.

2.1 Description of a physical line

Let us denote by L the set of points of the line. We do not orient the line. We choose one point O and we choose one of the two possible orientations of the line.

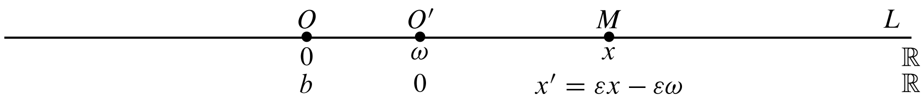


Now to each point M we can associate a real number x in a one to one map : you just measure the length OM in cm and put $+$ sign if OM is in the direction of the orientation you have choosen, $-$ if not. Let us call that map by c as "coordinate". Thus there is a bijection of L onto $\mathbb{R}$: $c : L \to \mathbb{R}, M \mapsto c(M) = x$.



In the above drawing $x = -4$, since our unit is the cm.

What happens if we choose another point O' and another orientation ? (The point O' may be equal to O or different, the new orientation chosen may be the same or the opposite orientation). We get another bijection (which can be the same if we have done the same choices as before !). Let us call it c' and put $x' = c'(M)$. Can we compute x' knowing x ? Of course we must say where we took the point O', that is we have to suppose that we know $c(O') = \omega$. We have also to say if the orientation is the same or opposite. Let us put $\varepsilon = +1$ whether the orientation is the same and $\varepsilon = -1$ if the orientation is opposite. Then $c'(M) = \varepsilon(c(M) - \omega)$, or $x' = \varepsilon x - \varepsilon\omega$.



The formula giving x'in term of x is the explicit form of the map $c' \circ c^{-1} : \mathbb{R} \to \mathbb{R}$. Let us put $b = -\varepsilon\omega$. We can summarize our study in the following way : we have a set L and a family of bijections $\mathcal{B}$ of L onto $\mathbb{R}$ such that if c and c' belong to $\mathcal{B}$, then there is a real number ε in $\{+1, -1\}$ and a real number b such that for all x in $\mathbb{R}$

$$(c' \circ c^{-1})(x) = \varepsilon x + b$$

2.2 Vectors

If we take two points A and B, we notice that the real number $c(B) - c(A)$ depends on the choice of the bijection c. This number is not intrinsic. We cannot define vectors as simply as in the case of the oriented line.

We can verify that the midpoint M of the couple of points (A, B) is intrinsic ; that is $c(M) = \frac{1}{2}(c(A) + c(B))$ if and only if for any $c' \in \mathcal{B}$ we have $c'(M) = \frac{1}{2}(c'(A) + c'(B))$. Put $x_A = c(A)$ and $x_B = c(B)$. We have to check that if $x_M = \frac{1}{2}(x_A + x_B)$ then for $x'_M = \varepsilon x_M + b, x'_A = \varepsilon x_A + b, x'_B = \varepsilon x_B + b$, we have $x'_M = \frac{1}{2}(x'_A + x'_B)$. That is easy mathematics !

Then we can define an equivalence relation $\equiv$ between couples of points by $(A, B) \equiv (C, D)$ if and only if $\{A, D\}$ and $\{B, C\}$ have the same midpoint.

Exercice 1. Verify that $\equiv$ is an equivalence relation.

We then define the vectors as the equivalence class of couples of points relative to the equivalence relation $\equiv$. Denote $\overrightarrow{L}$ the set of vectors :

$$\overrightarrow{L} = (L \times L)/ \equiv$$

It is easy to verify that $\overrightarrow{L}$ is isomorphic to F (in fact we have two isomorphisms).

2.3 Mathematical definitions

Do not forget everything this time. We keep the relation $\equiv$.

Definition. Let F be a field. An *Euclidean line over the field F* is a couple $(L, \mathcal{B})$, where L is a set and $\mathcal{B}$ is a set of bijections of L onto F such for any c and c' in $\mathcal{B}$ there is an element b in F and an element $\varepsilon \in \{1, -1\}$ such that

$$c' \circ c : F \to F, x \mapsto \varepsilon x + b$$

We say that the change of coordinates is given by

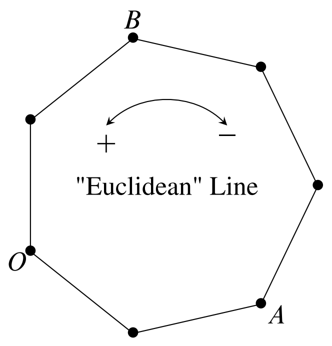
$$x' = \varepsilon x + b$$

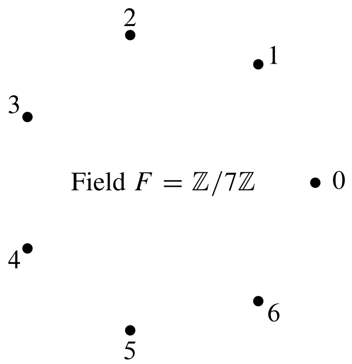
Remark. In any field F, the symbols 1 and -1 have a meaning. They are different unless $1 + 1 = 0$. One says that the characteristic of the field is 2. We suppose our Field is of characteristic different from 2 !

Example 1. Let L be the set of electric potential. It is an arbitrary choice that has been made to give the electron a negative charge. If we never change this convention the line of electric voltage is oriented, but... theoretically we could have done the opposite choice...

Example 2. Any line which is not necessarely a straight line, but such that one may measure distances along that line and without multiple points. The road may now be two-ways !

Example 3. Let us take the field $F = \mathbb{Z}/7\mathbb{Z}$. We may sketch the line in the following way





Exercice 2. In the above picture, if we choose the origin in O and the orientation $+$, what are the coordinates of A and B ? Describe the translation $T_{\overrightarrow{AB}}$.

2.4 Vectors

Proposition and Definition. Let $(L, \mathcal{B})$ be an oriented Euclidean line. The relation $\equiv$ defined in $L \times L$ by

$$(A, B) \equiv (C, D) \iff (A, D) \text{ and } (B, C) \text{ have same midpoint}$$

is an equivalence relation. The set $L \times L/\equiv$ is isomorphic to F and has thus the structure of a one-dimensional vector space on F.

Proof. The exercise 1 shows that $\equiv$ is an equivalence relation. Thus $\overrightarrow{L} = L \times L/\equiv$ is defined. Now choose any c in $\mathcal{B}$. Call O the point $c^{-1}(0)$. It is our origine. We have a bijection of $\overrightarrow{L} \to F, \overrightarrow{OM} \mapsto c(M)$, which gives $\overrightarrow{L}$ a structure of vector space independent of the choice of c. The change of orientation between c and c' corresponds to a change of the basis vector into its opposite. □

Proposition. The map $L \times L \to F,\ (A, B) \mapsto \overrightarrow{AB}$ has the following properties :

— for every point A, the map $L \to F,\ B \mapsto \overrightarrow{AB}$ is a bijection ;

— for all points A, B and C in L, we have $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Notation. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F. Letting $\overrightarrow{u}$ be an element of F, we denote by $M + \overrightarrow{u}$ the point N such that $\overrightarrow{MN} = \overrightarrow{u}$. In that case, we use also the notations $\overrightarrow{u} = N - M$.

Proposition and Definition. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F and let $\overrightarrow{u}$ be a vector in the vector space associated with L.

The map $T_{\overrightarrow{u}} : L \to L, M \mapsto M + \overrightarrow{u}$ is a bijection called *translation of vector* $\overrightarrow{u}$.

It is easy to check that the set of translations of $(L, \mathcal{B})$ is a group isomorphic to the group $(F, +)$ through

$$T_{\overrightarrow{u}} \circ T_{\overrightarrow{v}} = T_{\overrightarrow{u}+\overrightarrow{v}}$$

Proposition and Definition. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F and let Ω be a point in L. For any point M in L there is one and only one point N such that the midpoint of MN is Ω. The map $L \to L, M \mapsto N$ is called a central symmetry relative to Ω. Let us denote it by Sym_Ω.

Exercice 3. Show the following properties :

- $\mathrm{Sym}_\Omega \circ \mathrm{Sym}_\Omega = \mathrm{Id}_L$ where Id_L is the identity mapping on L ;
- $\mathrm{Sym}_\Omega \circ \mathrm{Sym}_{\Omega'} = T_{2\overrightarrow{\Omega'\Omega}}$;
- $T_{\overrightarrow{u}} \circ \mathrm{Sym}_\Omega = \mathrm{Sym}_{\Omega+\frac{1}{2}\overrightarrow{u}}$;
- $\mathrm{Sym}_\Omega \circ T_{\overrightarrow{u}} = \mathrm{Sym}_{\Omega-\frac{1}{2}\overrightarrow{u}}$.

Question : How does the complex line look like ?

§ 3. "Euclid's" 1-dimensional geometry

At the time of Euclid there was no common unit of length. In fact the only notion used was the ratios of lengths. Thus the concept of midpoint was clear.

3.1 Mathematical definitions

Definition. Let F be a field. A *line over the field* F is a couple $(L, \mathcal{B})$, where L is a set and $\mathcal{B}$ is a set of bijections of L onto F such for any c and c' in $\mathcal{B}$ there is an element b in F and an element a in F different from 0 such that

$$c' \circ c : F \to F, x \mapsto ax + b$$

We say that the change of coordinates is given by

$$x' = ax + b$$

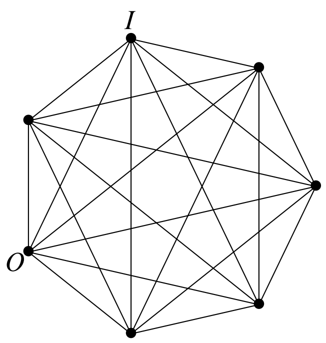
Remark. The function $x \mapsto ax$ is linear and the function $x \mapsto ax + b$ is affine.

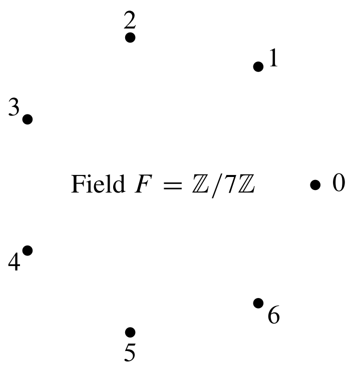
Example 1. Let L be the set of temperatures in "normal" life (I mean that we forget about Kelvin's absolute temperatures). The change of coordinates from °Celsius (denoted x) to °Fahrenheit (denoted x') is

$$x' = \frac{9}{5}x + 32$$

Example 2. Any line which is not necessary a straight line, but such that one may measure distances along that line and without multiple points. The road may now be two-ways !

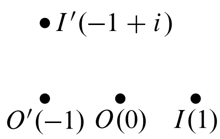
Example 3. Let us take the field $F = \mathbb{Z}/7\mathbb{Z}$. We may sketch the line in the following way





Exercice 4. In the above picture, we choose the origin in O and the point whose coordinate is 1 in I. Give the coordinates of the five other points.

Example 4. Let $(L, \mathcal{B})$ be a line on the field $\mathbb{C}$. Such a line is called a complex line (and often a complex plane because $\mathbb{C}$ is in bijection with $\mathbb{R} \times \mathbb{R}$). We take a new origin O' at the point such that $c(O') = -1$ and the point I' such that $c'(I') = 1$ at the point with coordinate $i - 1$. Compute $z' = c'(M)$ in terms of $z = c(M)$ for any point M.



3.2 Vectors

Proposition and Definition. Let $(L, \mathcal{B})$ be a line. The relation $\equiv$ defined in $L \times L$ by

$$(A, B) \equiv (C, D) \iff (A, D) \text{ and } (B, C) \text{ have same midpoint}$$

is an equivalence relation. The set $L \times L/\equiv$ is isomorphic to F and has thus the structure of a one-dimensional vector space on F.

Proof. Same as before. □

Proposition. The map $L \times L \to F$, $(A, B) \mapsto \overrightarrow{AB}$ has the following properties :

— for every point A, the map $L \to F$, $B \mapsto \overrightarrow{AB}$ is a bijection ;

— for all points A, B and C in L, we have $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Notation. Let $(L, \mathcal{B})$ be an oriented Euclidean line constructed on the field F. Letting $\overrightarrow{u}$ be an element of F, we denote by $M + \overrightarrow{u}$ the point N such that $\overrightarrow{MN} = \overrightarrow{u}$. In that case, we use also the notations $\overrightarrow{u} = N - M$.

Proposition and Definition. Let $(L, \mathcal{B})$ be a line constructed on the field F and let $\overrightarrow{u}$ be a vector in the vector space associated with L.

The map $T_{\overrightarrow{u}} : L \to L, M \mapsto M + \overrightarrow{u}$ is a bijection called *translation of vector* $\overrightarrow{u}$.

The set of translations of $(L, \mathcal{B})$ is still a group isomorphic to the group $(F, +)$.

Proposition and Definition. Let $(L, \mathcal{B})$ be a line constructed on the field F, let Ω be a point in L and let λ be an element of F. For any point M in L there is one and only one point N such that $\overrightarrow{\Omega N} = \lambda \overrightarrow{\Omega M}$. The map $L \to L, M \mapsto N$ is called a *homothety with center* Ω *and ratio* λ. Let us denote it by $\mathrm{H}_{\Omega,\lambda}$.

Exercice 5. Show that a central symmetry is a homothety. What is its ratio ?

Exercice 6. Let c be an element of $\mathcal{B}$, let $\Omega \in L$ and $\lambda \in F$. We call ω the coordinate $c(\Omega)$ of the point Ω. For any point M in L, we put $x = c(M)$ and $x' = c(\mathrm{H}_{\Omega,\lambda}(M))$. Give the formula for computing x' knowing x.

Exercice 7. The composition of two affine maps of F on itself is still an affine map. More precisely : if $g_1(x) = a_1 x + b_1$ and $g_2(x) = a_2 x + b_2$, then $g_1 \circ g_2 = g$ where $g(x) = ax + b$, $a = a_1 a_2$ and $b = a_1 b_2 + b_1$. Show that you find back these results using matrices :

$$\begin{bmatrix} a_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

3.3 Intrinsic objects

Anything defined on a line which is independent of the choice of coordinate is called intrinsic and has a geometrical meaning. The vectors associated with a line are all collinear. If $\overrightarrow{u}$ and $\overrightarrow{v}$ are such vectors we can find an element λ in F such that $\overrightarrow{v} = \lambda \overrightarrow{u}$. This number is the ratio of the two vectors. Let us take three points on a line A, B and C such that $C \neq A$, the ratio λ such that $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ is thus intrinsic. In France we have been using the notation $\frac{\overline{AB}}{\overline{AC}}$ for this ratio. We could also use $\frac{\overrightarrow{AB}}{\overrightarrow{AC}}$ but in higher dimensions that would be a tensor...

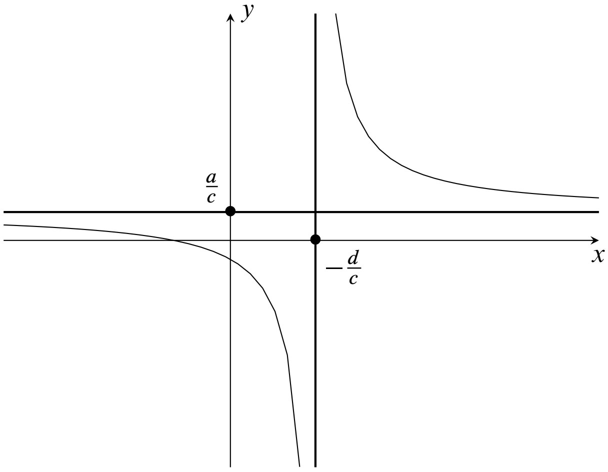
Theorem. The bijections of a line preserving ratios are the translations and the homothety.

§ 4. Projective 1-dimensional geometry

In the projective line, instead of using the group of affine bijections, we'll use the group of homographic functions

$$h(x) = \frac{ax+b}{cx+d} \qquad \text{where } ad - bc \neq 0$$

If $F = \mathbb{R}$, draw examples of homographic functions with Geogebra.

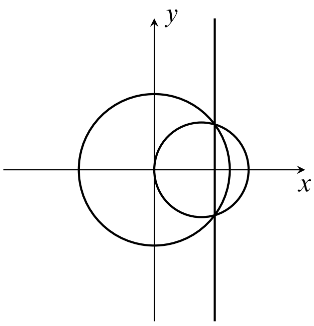


Notice the two asymptotes when $c \neq 0$. One is vertical (parallel to Oy) with equation $x = -d/c$ and one is horizontal (parallel to Ox) with equation $y = a/c$.

If $F = \mathbb{C}$ these functions are called *Möbius transformations*. The variable is usually denoted by z and here a, b, c and d are complex numbers. One simple example is

$$h(z) = \frac{1}{z}$$

The circle $|z| = 1$ has itself as image. The image of the line $\Re(z) = 0{,}8$ is the circle with radius 0,625 and center at $z_{\text{center}} = 0{,}625$.



4.1 The group of homographic functions on F

If you combine two homographic functions, you still get a homographic function. What are the computation rules ?

Let us associate the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to the above homography. We then denote the homography by h_A. The magic phenomenon is that

$$h_{A_1} \circ h_{A_2} = h_{A_1 A_2}$$

We have to do the computation completely at least one time :

$$(h_{\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}} \circ h_{\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}})x = \frac{a_1 \frac{a_2 x + b_2}{c_2 x + d_2} + b_1}{c_1 \frac{a_2 x + b_2}{c_2 x + d_2} + d_1} = \frac{(a_1 a_2 + b_1 c_2)x + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + c_2 d_1)x + (c_1 b_2 + d_1 d_2)}$$

and the last fraction is $h_A(x)$ for $A = A_1 A_2$.

The function h defined above $h(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ is a bijection of F onto F only when $c = 0$. If $c \neq 0$ then h is a bijection of $F \setminus \{-\frac{d}{c}\}$ onto $F \setminus \{\frac{a}{c}\}$.

How to get the homographies to become bijections ?

Answer : add one point, call it infinity and denote it ∞.

We shall still denote by h the bijection of $F \cup \{\infty\}$ onto itself by adding the two following rules :

$$h(\infty) = \frac{a}{c} \qquad \text{and} \qquad h(-\frac{d}{c}) = \infty$$

And finally we have to define if $c = 0$: $h(\infty) = \infty$.

Remark. What we usually call the complex plane is in fact a complex projective line. We have for instance the following nice theorem : The points corresponding to the complex numbers z_1, z_2, z_3 and z_4 are cocyclical if and only if the cross-ratio of these four numbers is real ("cocyclical" means on a same circle or a same line).

4.2 The projective line on a field F

Definition. A *real projective line* is a couple $(L, \mathcal{B})$ where L is a set and $\mathcal{B}$ is a set of bijections of L onto $\mathbb{R} \cup \{\infty\}$ such that for any two bijections c and c' belonging to $\mathcal{B}$ the composition of maps $c' \circ c$ is a homography.

Definition. The cross-ratio of four numbers x_1, x_2, x_3 and x_4 denoted $(x_1, x_2; x_3, x_4)$ is defined by

$$(x_1, x_2; x_3, x_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_4 - x_1)(x_3 - x_2)} = \frac{\frac{x_3 - x_1}{x_4 - x_1}}{\frac{x_3 - x_2}{x_4 - x_2}}$$

Theorem. Let A, B, C and D be four points on a projective line $(L, \mathcal{B})$. The cross-ratio of the coordinates of these four points in that order is independent of the choice of the bijection

c in $\mathcal{B}$. Thus the cross-ratio of four points may be defined by

$$(A, B; C, D) = (c(A), c(B); c(C), c(D)) = \frac{\overrightarrow{AC}\,\overrightarrow{BD}}{\overrightarrow{AD}\,\overrightarrow{BC}} = \frac{\frac{\overrightarrow{AC}}{\overrightarrow{AD}}}{\frac{\overrightarrow{BC}}{\overrightarrow{BD}}}$$

and it is an intrinsic quantity.

Proof. Put $x'_k = \frac{ax_k+b}{cx_k+d}$ for $k \in \{1, 2, 3, 4\}$. And compute

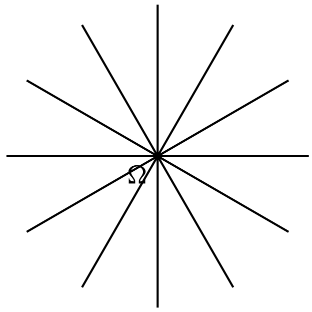
$$(x'_1, x'_2; x'_3, x'_4) = \frac{\frac{ax_3+b}{c_3+d} - \frac{ax_1+b}{cx_1+d}}{\frac{ax_4+b}{cx_4+d} - \frac{ax_2+b}{cx_2+d}} \cdot \frac{\frac{ax_4+b}{c_4+d} - \frac{ax_2+b}{cx_2+d}}{\frac{ax_3+b}{cx_3+d} - \frac{ax_2+b}{cx_2+d}} = (x_1, x_2; x_3, x_4) \qquad \square$$

Fundamental example of a real projective line.

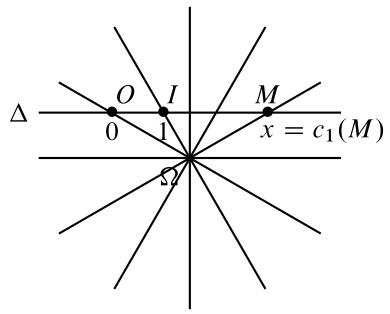
Let $\mathcal{E}$ be real Euclidean plane and Ω a point in $\mathcal{E}$. The set L of lines passing through Ω is a projective line.

Attention. Each LINE in $\mathcal{E}$ which passes through O is a POINT belonging to the line L.

Here below we have 6 points of the line L :



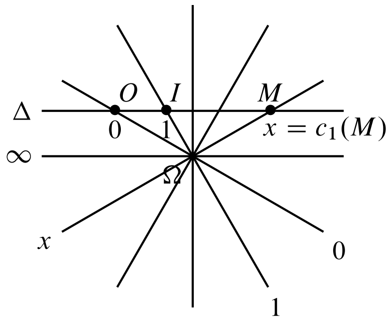
Until now L is just a set. Where and how are the coordinates ? Cut the above picture by any line Δ that does not go through Ω. That line Δ is an affine line. Let us chose two points O and I. We have one bijection $c_1 : \Delta \to \mathbb{R}$ such that $c_1(O) = 0$ and $c_1(I) = 1$.



Now we define the coordinates of the point of L in the following way

$$c(\text{line } \Omega O) = 0$$
$$c(\text{line } \Omega I) = 1$$
$$c(\text{line } \Omega M) = c_1(M) \quad \text{for } M \in \Delta$$

Now I hope you guess which line going through Ω will get the value ∞ (do not turn the page before answering !).



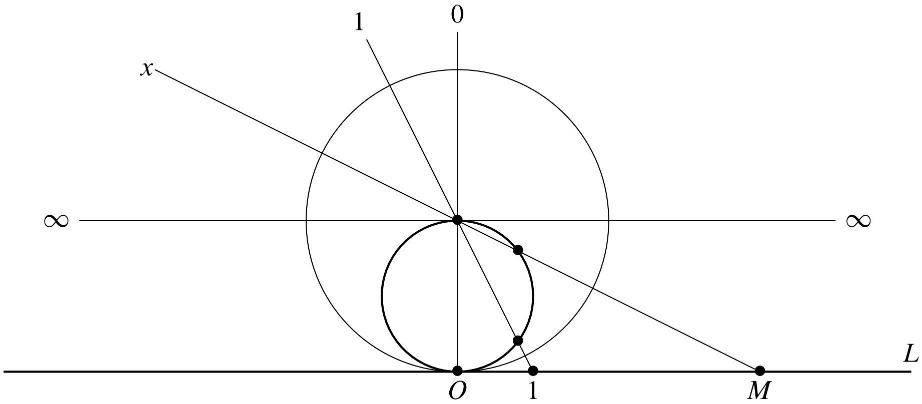
To end our construction we our going to use a theorem of basic plane geometry.

Theorem. The central projection of a line on an other one preserves the cross-ratios.

Thus if we take any other line Δ' we would get a coordinate c'_1 such that $c'_1 \circ c^{-1}$ is a real homography.

Another aspect of the real projective line

The real projective line is a circle !



Aspect of the complex projective line

Stereographic projection

Central projection

The complex projective line is called the Riemann Sphere !

4.3 Homogeneous coordinates

Instead of using a coordinate x we shall use two elements of F, let’s call them X and T and put

$$x = \frac{X}{T}$$

We put X and T in a column-matrix $\begin{bmatrix} X \\ T \end{bmatrix}$ and say that $\begin{bmatrix} X \\ T \end{bmatrix}$ and $\begin{bmatrix} X' \\ T' \end{bmatrix}$ describe the same point if they are colinear vectors in $\mathbb{R}^2$. The numbers X and T are called the *homogeneous coordinates* of the point on the projective line.

With this "trick" we see that the homographic function h_A may simply be written

$$\begin{bmatrix} X' \\ T' \end{bmatrix} = A \begin{bmatrix} X \\ T \end{bmatrix}$$

But this is nice not only because we have transformed homographic functions into linear function, but also because with this new formalisme, we can include easily the "point at infinity". And furthermore in higher dimensions it helps to clarify the relations between all the points at infinity.

Chapitre 5

General definition of a geometry

https ://people.maths.ox.ac.uk/hitchin/hitchinnotes/Projective_geometry/Chapter_4_The_Klein_programme.pdf

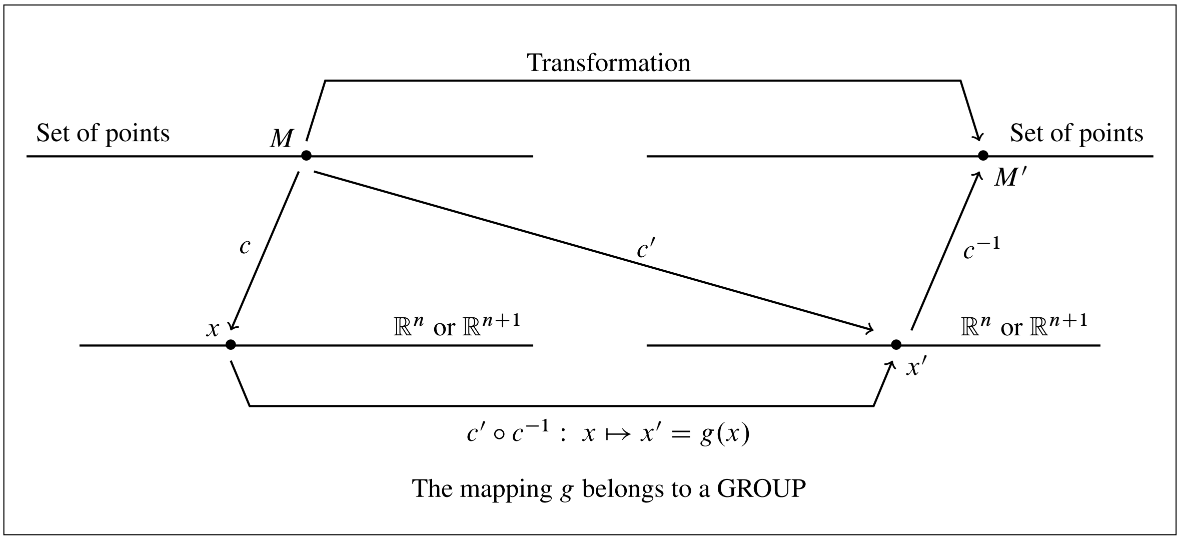
Nigel Hitchin (Savilian Professor of Geometry, University of Oxford) writes :

Felix Klein prepared for his inaugural address in 1872 in Erlangen a paper which gave a very general view on what geometry should be regarded as. It was somewhat controversial at the time and in fact he spoke on something different for his lecture, but the point of view is still called the Erlanger Programm. Klein saw geometry as :

the study of invariants under a group of transformations.

This throws the emphasis on the group rather than the space, and was highly influential in a number of ways.

§ 1. The general setting



Practically, the important thing is the formula hidden in g.

$\boldsymbol{n = 1}$

- $x' = x + b$: Euclidean oriented line
- $x' = \varepsilon x + b$: Euclidean line
- $x' = ax + b,\ a \neq 0$: "the line of Euclid" if $F = \mathbb{R}$
- $x' = \frac{ax+b}{cx+d},\ ad - bc \neq 0$: the projective line
- $\begin{bmatrix} X' \\ T' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ T \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible : still the projective line

$\boldsymbol{n \geqslant 1}$

- $\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$: group of translations
- $\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = \varepsilon \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$: group of translations and central symmetries
- $\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ where $AA^T = A^T A = I$, det$A = +1$: The direct Euclidean isometry group, or the special Euclidean group, whose elements are called Euclidean motions, displacements or rigid motions.
- $\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ where $AA^T = A^T A = I$: The Euclidean group E(n), also known as ISO(n) or similar, is the symmetry group of n-dimensional Euclidean space. Its elements are the isometries associated with the Euclidean distance, and are called Euclidean isometries, Euclidean transformations or Rigid transformations..
- $\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ where det $A \neq 0$: The affine group
- $\begin{bmatrix} X'_1 \\ X'_2 \\ \vdots \\ X'_n \\ T' \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ T \end{bmatrix}$ where det $A \neq 0$: The projective group

§ 2. An example : the real planes

I begin with some repetitions...

2.1 Permutations of the four points of a cross-ratio

In the following paragraph, I use the following notation : if A and B are two points of projective line $(L, \mathcal{B})$ and if c is a coordinate belonging to $\mathcal{B}$

$$AB = c(B) - c(A)$$

Notice that with that notation AB can be negative and the length of the segment with end-points A and B is $\left|AB\right|$.

Then the definition of the cross-ratio becomes

$$(A, B; C, D) = \frac{AC}{AD}\frac{BD}{BC} = \frac{AC \cdot BD}{AD \cdot BC}$$

Let us call r the cross-ratio $(A, B; C, D)$

$$(A, B; C, D) = r$$

If we exchange the couples (A, B) and (C, D) nothing is changed. Thus $(C, D; A, B) = r$.

If we exchange A and B, we get $(B, A; C, D) = \frac{BC \cdot AD}{BD \cdot AC} = \frac{1}{\frac{AC \cdot BD}{AD \cdot BC}} = \frac{1}{r}$.

Using the relation above we get $(D, C; A, B) = (A, B; D, C) = \frac{1}{r}$. Exchanging A and B again, we get $(B, A; D, C) = r$. Thus

$$\boxed{r = (A, B; C, D) = (B, A; D, C) = (C, D; A, B) = (D, C; B, A)}$$

and

$$\boxed{\tfrac{1}{r} = (A, B; D, C) = (B, A; C, D) = (D, C; A, B) = (C, D; B, A)}$$

Let's compute

$$\begin{aligned}(A, C; B, D) = \frac{AB \cdot CD}{AD \cdot CB} = \frac{(AD + DB)(CB + BD)}{AD \cdot CB} &= 1 + \frac{DB(CB + BD - AD)}{AD \cdot CB} \\ = 1 + \frac{DB(CB + BD - AD)}{AD \cdot CB} &= 1 - r\end{aligned}$$

Making use of the two relations above, we get

$$\boxed{1 - r = (A, C; B, D) = (C, A; D, B) = (B, D; A, C) = (D, B; C, A)}$$

and

$$\boxed{\tfrac{1}{1-r} = (A, C; D, B) = (C, A; B, D) = (D, B; A, C) = (B, D; C, A)}$$

Notice that $1 - \frac{1}{1-r} = \frac{r}{r-1}$. Exchanging the letters in second and third position in the preceding formula, we get

$$\boxed{\frac{r}{r-1} = (A, D; C, B) = (D, A; B, C) = (C, B; A, D) = (B, C; D, A)}$$

The inverse of $\frac{r}{r-1}$ is $1 - \frac{1}{r}$, thus

$$\boxed{1 - \frac{1}{r} = (A, D; B, C) = (D, A; C, B) = (C, B; D, A) = (B, C; A, D)}$$

Question : Doing a permuation of the points A, B, C et D we get 6 values : r, $\frac{1}{r}$, $1-r$, $\frac{1}{1-r}$, $\frac{r}{r-1}$ and $1 - \frac{1}{r}$. For which values of r do some of these values be equal ? There are three families of answers :

1. If one of the values belongs to $\{0, 1, \infty\}$, then the 5 others also (each value is found 2 times)
2. If one of the values belongs to $\{-1, \frac{1}{2}, 2\}$, then the 5 others also (each value is found 2 times)
3. If one of the values belongs to $\{\frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}\}$, then the 5 others also (each value is found 3 times)

The first case happens when two of the four points are the same. The third case is interesting for the complex projective line. We are left with the second case.

Definition. Let A, B, C and D be four points on a line. They are said to form a *harmonic range* if $(A, B; C, D) = -1$. In this case, we also say that D is the *harmonic conjugate* of C relative to A and B.

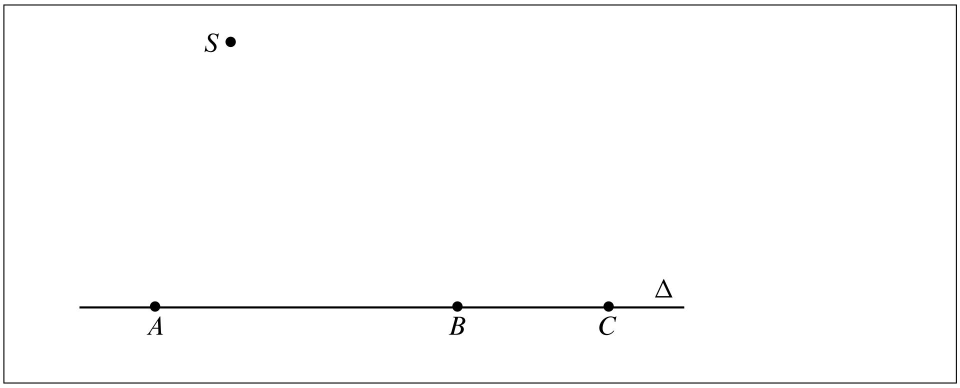
Definition. Let Δ and Δ' be two lines in a projective plane. Let S be a point of the plane which does not belong to Δ neither to Δ'. The central projection of Δ on Δ' with center S is the map $p : \Delta \to \Delta', M \mapsto M'$, where M' is the intersection point of the lines SM and Δ'.

Theorem. Central projection of a line on another preserves cross-ratios. If the lines are projective lines the central projections are bijection.

Corollary. Let Δ and Δ' be two lines in a projective plane. Let S be a point of the plane which does not belong to Δ neither to Δ' and denote by p the central projection of Δ onto Δ'. If $c : \Delta \to \mathbb{R} \cup \{\infty\}$ is a coordinate on Δ then $c \circ p^{-1}$ is a coordinate on Δ'.

2.2 How to construct a harmonic conjugate ?

The answer :



Exercice 1. Check by computation that the drawing you have done has given the right point.

Exercice 2. Show that the method used above always gives the right answer.

Exercice 3. Given two lines which meet in a point K outside the sheet of paper and a point A between the two lines. How to draw the line AK (just a segment of that line which is inside the sheet) ?

** **Exercice 4.** You have a pen and a 5 cm long ruler. Given two points A and B at a distance about 15 cm, how to draw the line AB ?

2.3 Homogeneous coordinates

Put $x = \frac{X}{T}$ and $y = \frac{Y}{T}$. The equation of a line becomes

$$aX + bY + cT = 0$$

instead of $ax + by + c = 0$. Then we can write the equation of a line in the matrix form

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ T \end{bmatrix} = 0$$

which gives you immidiately the duality between points and lines...

Exercise Show the converse of the theorem of Desargues.

2.4 Pappus's theorem

Theorem. Let Δ and Δ' be two lines, let A, B and C be three points belonging to Δ and A', B' and C' be three points belonging to Δ'. Put $P = BC' \cap B'C$, $Q = CA' \cap C'A$ and $R = AB' \cap A'B$. The points P, Q and R are collinear.

Proof. Choose two points and decide that they are on the line at infinity. Redraw the picture and use parallel projection. □