Analysis IV
Spring 2011
Exercises 1 / Answers

(1) Let $\{x_1, \dots, x_n\}$ be a linearly independent set of vectors in the space $\mathbb{R}^n$. Prove that every $x \in \mathbb{R}^n$ has at most one representation of the form $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

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Suppose there are two different sets of scalars $\lambda_1, \dots, \lambda_n$ and $\lambda'_1, \dots, \lambda'_n$ so that

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

and

$$x = \lambda'_1 x_1 + \cdots + \lambda'_n x_n.$$

Then

$$0 = x - x = (\lambda_1 - \lambda'_1)x_1 + \cdots + (\lambda_n - \lambda'_n)x_n,$$

and as the set of $x_1, \dots, x_n$ is linearly independent, this implies that $\lambda_i - \lambda'_i = 0$ for each $i = 1, \dots, n$, that is, $\lambda_i = \lambda'_i$ for each $i = 1, \dots, n$. That is a contradiction, because the λ and λ' scalars were supposed to be different.

(2) Prove the linear subspace test (Theorem 1.3).

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> **Theorem 1.3 (Subspace test)** Let V a vector space (over $\mathbb{F}$) and let $U \subset V$ be a non-empty set. Then U is a linear subspace of V if and only if $\alpha x + \beta y \in U$ for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in U$.

Note this is a if-and-only-if (*jos-ja-vain-jos*) assertion; we need to prove it both ways.

First ("$\Rightarrow$"), suppose U is a linear subspace of V. This was defined to mean U is a vector space; and thus the scalar multiplication of elements of U stays in U, that is, since $x \in U$, we have $\alpha u \in U$, and vector addition of elements of U stays in U: since $\alpha x, \beta y \in U$, we have $\alpha x + \beta y \in U$.

Second ("$\Leftarrow$"), suppose $\alpha x + \beta y \in U$ for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in U$. To prove that U, with the same multiplication and addition as V, is a vector space, we need to go through the list of vector space properties. Most of these are clear, since if a property hold for all members of V, it holds for all members of $U \subset V$. The differences are those properties which are about something being included in U:

1) First, do addition and multiplication stay in U? Yes, because that is what our assumption of $\alpha x + \beta y \in U$ *means.* Addition stays in U because with $\alpha = \beta = 1$ we have $x + y \in U$. Multiplication stays in U because with $\beta = 0$ we have $\alpha x \in U$.

2) Is there a zero element in U? Since $\alpha x + \beta y \in U$ for all α, β, x, y, we can choose $\alpha = \beta = 0$, and since any vector multiplied with the zero scalar gives the zero vector (of V), we have

$$0x + 0y = 0 + 0 = 0 \in U.$$

Thus U has a zero element, and it is the same as the zero element of V.

3) Are the inverse elements (of $x \in U$) in U? Multiplication stays in U, so $(-1)x \in U$. Since

$$x + (-1)x = (1 + (-1))x = 0x = 0$$

(all these properties hold in V, so they hold in U), each $x \in U$ has an inverse element $-x$.

(3) Show that the set $F(S, V)$ in Definition 1.5 equipped with addition and scalar multiplication is a vector space.

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$F(S, V)$ denotes all functions $f : S \to V$, where S is some set (*joukko*) and V is a vector space (*vektoriavaruus*). For any scalar $\alpha \in V$ and any $f, g \in F(S, V)$, we define addition (*yhteenlasku*) and multiplication (*kertolasku*) as

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

To be really dense about it, this means that when we write $f+g$ or $(f + g)(x)$ we mean a new function whose value at point x is defined as $f(x) + g(x)$, or the values of functions f and g at x added to each other with the normal addition of V.

Now, a vector space is such a collection of scalars (here $\mathbb{F}$) and points (here the functions of $F(S, V)$), with some operations called "addition" (denoted with $+$) and "multiplication" (denoted with $\cdot$ or not at all), that for these the following hold: for any $\alpha, \beta \in \mathbb{F}$, and $x, y, z \in F(S, V)$,

(a) $f + g = g + f$ (commutativity / *vaihdannaisuus*)
(b) there exists a unique zero element (*nolla-alkio*) $0 \in F(S, V)$ such that $f + 0 = f$ for all $f \in F(S, V)$
(c) there exists a unique inverse element (*käänteisalkio*) $-f \in F(S, V)$ such that $f + (-f) = 0$ for all $f \in F(S, V)$
(d) $1 \cdot f = f$ and $\alpha(\beta f)) = (\alpha\beta)f$ (associativity / *liitännäisyys*)
(e) $\alpha(f + g) = \alpha f + \alpha g$ and $(\alpha + \beta)f = \alpha f + \beta f$ (distributivity / *osittelulait*)

Because V is a vector space, the whole list of vector space properties holds for any points in V. Since the points of $F(S, V)$ are functions whose values are in V, the vector space properties hold for any individual values of those functions, and hence for them as a whole. An example follows. Let $x \in S$ be fixed. If $\alpha, \beta \in \mathbb{F}$ and $f(x) \in V$, then $(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x)$, since V is a vector space. Since this holds for all $x \in S$, the equivalent property holds for $\alpha, \beta \in \mathbb{F}$ and $f \in F(S, V)$.

This means the vector space properties (a), (d) and (e) are clear.

Property (b), the zero element, has been given in the lectures.

Property (c), the inverse element, follows since for any fixed $x \in S$, $f(x) \in V$ has a (V-space) inverse element $-f(x)$, and those elements over all the possible values of x give the ($F(S, V)$-space) inverse element $-f$.

That is all.

(4) Prove that if S is the set of integers $\{1, \ldots, k\}$, then the set $F(S, \mathbb{F})$ can be identified with the space $\mathbb{F}^k$.

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Now $F(S, \mathbb{F})$ is the set of all functions $f : S \to \mathbb{F}$; it contains all the ways of connecting each of the k members of S into a member of $\mathbb{F}$. ($\mathbb{F}$ is either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers).) The set of all such mappings is a sequence of any k members of $\mathbb{F}$, including multiples; that is, $\mathbb{F}^k$.

To illustrate this, let $k = 3$, and let $\mathbb{F} = \mathbb{R}$. Then one function in $F(S, \mathbb{F})$ would be defined by $f(1, 2, 3) = (0, 42, 23)$; another might be $g(1, 2, 3) = (6, 6, 11)$, and a third $h(1, 2, 3) = (0, 0, 7)$. If we think of the three members of S as meaning ”this points at the first element”, ”this points at the second element”, and the third, the analogue to $\mathbb{R}^3$ is obvious.

(5) Show that the mapping $\mathbb{F} : \mathbb{R}^3 \to \mathbb{R}^2$,

$$F(x, y, z) = (y + z, x),$$

is linear, that is, that $F \in L(\mathbb{R}^3, \mathbb{R}^2)$.

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A mapping T is linear if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. If this F is linear, we should have the value of F at $\alpha(x, y, z) + \beta(a, b, c)$ be equal to α times $F(x, y, z)$ plus β times $F(a, b, c)$; that is,

$$\begin{aligned} &F(\alpha x + \beta a, \alpha y + \beta b, \alpha z + \beta c) \\ &= \alpha F(x, y, z) + \beta F(a, b, c). \end{aligned}$$

Here by the definition of F the left-hand side is

$$(\alpha y + \beta b + \alpha z + \beta c, \alpha x + \beta a),$$

while the right-hand side is

$$\begin{aligned} &\alpha(y + z, x) + \beta(b + c, a) \\ =&(\alpha y + \alpha z, \alpha x) + (\beta b + \beta c, \beta a) \\ =&(\alpha y + \alpha z + \beta b + \beta c, \alpha x + \beta a), \end{aligned}$$

which indeed are the same vector.

(6) Prove the Schwarz inequality: If $a_j, b_j \in \mathbb{F}$ for all $j = 1, \ldots, k$, then

$$\left(\sum_{j=1}^{k} |a_j||b_j|\right)^2 \leq \sum_{j=1}^{k} |a_j|^2 \cdot \sum_{j=1}^{k} |b_j|^2.$$

$$* * *$$

(There are many different proofs for this.)
Let us examine the sum

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_i||b_j| - |a_j||b_i|)^2.$$

First, because it is a sum of squares, it is non-negative. Second, it can be written as

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - 2|a_i||a_j||b_i||b_j|)$$

$$=\frac{1}{2}\left(\sum_{i=1}^{n}|a_i|^2\sum_{j=1}^{n}|b_j|^2 + \sum_{j=1}^{n}|a_j|^2\sum_{i=1}^{n}|b_i|^2 \right.$$
$$\left. -2\sum_{i=1}^{n}|a_i||b_i|\sum_{j=1}^{n}|a_j||b_j|\right).$$

This in turn can be written as

$$\sum_{j=1}^{k} |a_j|^2 \cdot \sum_{j=1}^{k} |b_j|^2 - \left(\sum_{j=1}^{k} |a_j||b_j|\right)^2.$$

Thus

$$\sum_{j=1}^{k} |a_j|^2 \cdot \sum_{j=1}^{k} |b_j|^2 - \left(\sum_{j=1}^{k} |a_j||b_j|\right)^2 \geq 0,$$

from which the claim follows.