Analysis IV
Spring 2011
Exercises 2

(1) Let ℓ^1 be the set of those infinite sequences $x = \{x_1, x_2, \dots\}$ with $x_n \in \mathbb{C}$ which satisfy the condition
$$\sum_{n=1}^{\infty} |x_n| < \infty.$$
Show that ℓ^1 equipped with addition $x+y = \{x_1+y_1, x_2+y_2, \dots\}$ and scalar multiplication $\alpha x = \{\alpha x_1, \alpha x_2, \dots\}$, $\alpha \in \mathbb{C}$, is a vector space over $\mathbb{C}$.

(2) Prove Theorem 1.13 (b) and (c).

(3) Show that the function $d : \ell^1 \times \ell^1 \to \mathbb{R}$,
$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|$$
is a metric. (When we are talking of ℓ^1 or some other sequences, the following four notations all mean the same, and are used somewhat randomly: x, $\{x_n\}$, $\{x_n\}_{n=1}^{\infty}$, $\{x_1, x_2, \dots\}$.)

(4) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric space (M, d). Prove that there exists $R > 0$ such that $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

(5) Let $\{a_n\}$ be a Cauchy sequence in the metric space (M, d). Prove: If the sequence $\{a_n\}$ has a subsequence which converges to $a \in M$, then $\{a_n\}$ converges to a.