Analysis IV
Spring 2011
Exercises 2 / Answers

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(1) Let ℓ^1 be the set of those infinite sequences $x = \{x_1, x_2, \ldots\}$ with $x_n \in \mathbb{C}$ which satisfy the condition

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

Show that ℓ^1 equipped with addition $x+y = \{x_1+y_1, x_2+y_2, \ldots\}$ and scalar multiplication $\alpha x = \{\alpha x_1, \alpha x_2, \ldots\}$, $\alpha \in \mathbb{C}$, is a vector space over $\mathbb{C}$.

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The same thing as last week: Let x, y and z be in ℓ^1, where $x = \{x_1, x_2, \ldots\}$ and y and z likewise. Let $\alpha, \beta \in \mathbb{C}$.

(a) Addition is commutative, since

$$\begin{aligned} x + y &= \{x_1 + y_1, x_2 + y_2, \ldots\} \\ &= \{y_1 + x_1, y_2 + x_2, \ldots\} = y + x, \end{aligned}$$

and it is associative[1], since

$$\begin{aligned} x + (y + z) &= x + \{y_1 + z_1, y_2 + z_2, \ldots\} \\ &= \{x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots\} \\ &= \{x_1 + y_1, x_2 + y_2, \ldots\} + z = (x + y) + z. \end{aligned}$$

(b) There is a unique zero element, $0 = \{0, 0, \ldots\}$, since for any $x \in \ell^1$,

$$x + 0 = \{x_1 + 0, x_2 + 0, \ldots\} = \{x_1, x_2, \ldots\} = x.$$

(c) For each $x \in \ell^1$ there is an inverse element $-x$, defined by

$$-x = \{-x_1, -x_2, \ldots\},$$

since

$$x + (-x) = \{x_1 - x_1, x_2 - x_2, \ldots\} = \{0, 0, \ldots\} = 0.$$

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[1]A random bit of motivation. Many operations are associative, but not all. For the normal multiplication of real numbers we have $a(bc) = (ab)c$, but for their division we most certainly don't have $a/(b/c) = (a/b)/c$.

Also, this $-x$ is in the space ℓ^1, since

$$\sum_{n=1}^{\infty} |-x_n| = \sum_{n=1}^{\infty} |x_n| < \infty.$$

(d) Clearly $1 \cdot x = x$, since

$$1 \cdot x = \{1 \cdot x_1, 1 \cdot x_2, \ldots\}.$$

Likewise, $\alpha(\beta x) = (\alpha\beta)x$.

(e) Clearly

$$\begin{aligned} \alpha(x+y) &= \alpha(\{x_1+y_1, x_2+y_2, \ldots\}) \\ &= \{\alpha(x_1+y_1), \alpha(x_2+y_2), \ldots\} \end{aligned}$$

and

$$\alpha x + \alpha y = \{\alpha x_1, \alpha x_2, \ldots\} + \{\alpha y_1, \alpha y_2, \ldots\},$$

which are the same. Likewise with $(\alpha+\beta)x = \alpha x + \beta x$.

(2) Prove Theorem 1.13 (b) and (c).

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Theorem 1.13 Suppose that $\{x_n\}$ is a convergent sequence (with a limit x) in a metric space (M, d). Then:

(b) any subsequence of $\{x_n\}$ also converges to x,

(c) $\{x_n\}$ is a Cauchy sequence.

Proof: (b) Let $\epsilon > 0$. Let $\{x_{n_j}\} \subset \{x_n\}$. Because $\{x_n\}$ converges to x, there is a $N_\epsilon \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N_\epsilon$. Because $\{x_{n_j}\}$ is a subsequence of x_n, there is a $j_N \in \mathbb{N}$ such that $n_j > N$ for all $j > j_N$, and, consequently, $d(x_{n_j}, x) < \epsilon$.

(c) Let $\epsilon > 0$. Since $\{x_n\}$ converges to x, there is a $N \in \mathbb{N}$ so that

$$d(x_n, x) < \epsilon/2$$

for all $n \geq N$. Then, for all $n, m \geq N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

(3) Show that the function $d : \ell^1 \times \ell^1 \to \mathbb{R}$,

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|$$

is a metric. (When we are talking of ℓ^1 or some other sequences, the following four notations all mean the same, and are used somewhat randomly: x, $\{x_n\}$, $\{x_n\}_{n=1}^{\infty}$, $\{x_1, x_2, \ldots\}$.)

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A function d is a metric, if the following four properties hold:

(a) $d(x,y) \geq 0$
(b) $d(x,y) = 0 \Leftrightarrow x = y$
(c) $d(x,y) = d(y,x)$
(d) $d(x,z) \leq d(x,y) + d(y,z)$ (the triangle inequality)

For our function d, the properties are rather obvious —

(a) $d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|$, which as a sum of non-negative terms is also non-negative.
(b) If $d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n| = 0$, then $|x_n - y_n| = 0$ for every n, because the terms of the sum are all non-negative. This is possible only when $x_n = y_n$ for every n, that is, when $\{x_n\} = \{y_n\}$. The converse is obvious.
(c) $d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n| = \sum_{n=1}^{\infty} |y_n - x_n| = d(\{y_n\},\{x_n\})$ by the same property of the common absolute value.
(d) For any n, we have

$$\begin{aligned} &|x_n - z_n| = |x_n - y_n + y_n - z_n| \\ \leq &|x_n - y_n| + |y_n - z_n|. \end{aligned}$$

The triangle inequality follows by induction.

(4) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric space (M,d). Prove that there exists $R > 0$ such that $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

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Since $\{x_n\}$ is a Cauchy sequence, there is a $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $m, n \geq N_1$, and especially

$$d(x_{N_1}, x_m) < 1$$

for all $m \geq N_1$. We now set $M_1 := d(x_1, x_{N_1}) + 1$, and note that by the triangle inequality, for all $n \geq N_1$, we have

$$d(x_1, x_n) \leq d(x_1, x_{N_1}) + d(x_{N_1}, x_n) < d(x_1, x_{N_1}) + 1 = M_1.$$

Since there are $N_1 - 1$ points in the sequence before x_{N_1}, we have

$$M_2 := \max_{1 \leq n \leq N_1 - 1} d(x_1, x_n) < \infty.$$

Thus, for all points x_n, the following holds:

$$d(x_1, x_n) < \max(M_1, M_2).$$

This is the same as saying that all points x_n are in a x_1-centric ball of radius $R := \max(M_1, M_2)$, or $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

(5) Let $\{a_n\}$ be a Cauchy sequence in the metric space (M, d). Prove: If the sequence $\{a_n\}$ has a subsequence which converges to $a \in M$, then $\{a_n\}$ converges to a.

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Let $\{a_{n_j}\}$ be the convergent subsequence of $\{a_n\}$, and let $\epsilon > 0$. We will now show that there is a $N_\epsilon \in \mathbb{N}$ such that

$$d(a_n, a) < \epsilon$$

for all $n \geq N_\epsilon$.

Since $\{a_n\}$ is a Cauchy sequence, there is a $N_2 \in \mathbb{N}$ such that $d(a_n, a_m) < \epsilon/2$ when $n, m \geq N_2$.

Since $\{a_{n_j}\}$ converges to a, there is a $N_1 \in \mathbb{N}$ such that $a_{N_1} \in \{a_{n_j}\}$, $d(a_{N_1}, a) < \epsilon/2$ and $N_1 \geq N_2$.

Thus, for any $n > N =: \max(N_1, N_2)$, we have by the triangle inequality that

$$d(a_n, a) \leq d(a_n, a_{N_1}) + a(a_{N_1}, a) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$