Analysis IV
Spring 2011
Exercises 3 / Answers

(1) Let (E, d) be a metric space and let $x \in E$, $A \subset E$. Define
$$d(x, A) = \inf_{y \in A} d(x, y).$$
Show that $\{x \mid d(x, A) = 0\} = \overline{A}$.

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The closure (*sulkeuma*) of A is denoted by $\overline{A}$ and is defined as the smallest closed set which contains A. We know that $A \subset \overline{A}$.

$\boxed{\text{"}\subset\text{"}}$: Let $x \in E$ so that $d(x, A) = 0$ and $x \notin \overline{A}$. Because $\overline{A}$ is closed, $E \setminus \overline{A}$ is open, and $x \in E \setminus \overline{A}$. Thus there is a radius $r > 0$ so that $x \in B(x, r) \subset E \setminus \overline{A}$. This means $d(x, A) \geq r$, which is a contradiction. Hence $\{x \mid d(x, A) = 0\} \subset \overline{A}$.

$\boxed{\text{"}\supset\text{"}}$: Let $x \in E$ so that $x \in \overline{A}$ and $x \notin \{x \mid d(x, A) = 0\}$. Then $d(x, A) = 2r > 0$ for some $r > 0$. Since $B(x, r)$ is open, $E \setminus B(x, r)$ is closed. Since for each $y \in A$ we have
$$d(x, y) \geq d(x, A) = 2r > r,$$
we have $A \subset E \setminus B(x, r)$. Because $E \setminus B(x, r)$ is a closed set that contains A, and it does not contain x, and since $\overline{A}$ is the smallest closed set that contains A, we have $x \notin \overline{A}$. This is a contradiction; hence $\{x \mid d(x, A) = 0\} \supset \overline{A}$.

(2) Let X be an infinite set. Let $\mathbb{T}$ consist of $\emptyset$, X and all sets G such that $X \setminus G$ is a finite set. Prove that $(X, \mathbb{T})$ is a topological space.

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We know that $(X, \mathbb{T})$ is a topological space if:

(a) $\boxed{\emptyset \in \mathbb{T} \text{ and } X \in \mathbb{T}.}$ (This is okay.)

(b) $\boxed{\text{the union of sets stays in } \mathbb{T}.}$

Let $G_i \in \mathbb{T}$, $i = 1, 2, \ldots$. Then $X \setminus G_i$ is finite for every G_i, and thus
$$X \setminus \left(\bigcup_i G_i \right) = \bigcap_i (X \setminus G_i)$$
is finite too, being an intersection of finite sets.

(c) $\boxed{\text{the intersection of a pair of sets stays in } \mathbb{T}.}$

If $X \setminus G_1$ and $X \setminus G_2$ are finite for some $G_1, G_2 \in \mathbb{T}$, then
$$X \setminus (G_1 \cap G_2) = (X \setminus G_1) \cup (X \setminus G_2)$$
is, as the union of two finite sets, also finite.

(3) Let $A \subset \mathbb{R}^n$ be a set whose every point has a neighborhood which includes only a countable number of points of A. Prove that A is countable. (Hint: Lindelöf's covering theorem)

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Lindelöf's covering theorem : Every open covering of a set $A \subset \mathbb{R}^n$ has a countable subcovering.

We have for each $x \in A$ a neighborhood B_x so that B_x contains only a countable number of points of A. Let T be the set of all sets B_x. Then T is a covering of A. As neighborhoods are open sets, T is an open covering of A, and by Lindelöf it has a countable subcovering. Let the sets of this subcovering be T_i, $i = 1, 2, 3, \ldots$. Since each set T_i has only a countable number of points of A, we can denote those points by $t_{i,j}$, $j = 1, 2, 3, \ldots$. Since

$$A = \bigcup_{i,j \,\in\, \mathbb{N}} \{t_{i,j}\},$$

A is countable.

(4) Prove that a collection of disjoint open sets in $\mathbb{R}^n$ is either finite or countable.

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Let G be a (not necessarily countable) collection of disjoint open sets $G_i \subset \mathbb{R}^n$. Then G is an open covering of $\cup_i G_i$, and by Lindelöf's covering theorem G has a countable subcovering. But since G_i are disjoint, the only possible "subcovering" is G itself, so G is (at most) countable.

(5) Let f be a continuous real function on a metric space X. Let $\mathbb{Z}(f)$ be the set of all $p \in X$ for which $f(p) = 0$. Prove that $\mathbb{Z}(f)$ is closed.

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By definition, $\mathbb{Z}(f)$ is closed if its complement $X \setminus \mathbb{Z}(f)$ is open. Let $p \in X \setminus \mathbb{Z}(f)$. Then $f(p) \neq 0$. Because f is continuous, there is a $\delta > 0$ so that

$$|f(p) - f(y)| < f(p)/2 \qquad (1)$$

when $d(y, p) < \delta$. From (1) we get $f(y) \neq 0$ if $d(y, p) < \delta$. Since any $p \in X \setminus \mathbb{Z}(f)$ thus has an environment $B_d(p, \delta) \subset X \setminus \mathbb{Z}(f)$, we have that $X \setminus \mathbb{Z}(f)$ is open, and thus $\mathbb{Z}(f)$ is closed.