Analysis IV
Spring 2011
Exercises 4 / Answers

---

(1) Let $0 < a < 1$, and assume it is known that the functions $f_n(x) = nx(1-x)^n$ converge to $f(x) = 0$ as $n \to \infty$. Is the convergence uniform on $[a, 1]$? What about $[0, 1]$?

$* * *$

We say that f_n converges uniformly to f if

$$\sup\{|f_n(x) - f(x)| \mid x \in M\} \to 0 \quad \text{as } n \to \infty.$$

Here we have the cases $M = [a, 1]$, $0 < a < 1$, and $M = [0, 1]$ to consider. For both, we begin with writing out $|f_n(x) - f(x)|$ and finding its extremal values (*ääriarvot*).

First,

$$|f_n(x) - f(x)| = nx(1-x)^n - 0,$$

and since this is a continuous function, we know it gets its largest values either at the zeros of its derivative (*derivaatan nollakohdissa*) or at the endpoints of M. The derivative is

$$D(nx(1-x)^n) = n(1-x)^n - n^2x(1-x)^{n-1},$$

and setting $n(1-x)^n - n^2x(1-x)^{n-1} = 0$ we get, after some ~~boring~~ *interesting* calculation, $x = 1/(1+n)$.

a) First, let $M = [a, 1]$, $0 < a < 1$.

The possible points for the extremal values of $|f_n(x) - f(x)|$ are $x = 1/(1+n)$, $x = a$ and $x = 1$. At these points we have

$$\begin{aligned} |f_n(a) - f(a)| &= na(1-a)^n, \\ |f_n(1) - f(1)| &= 0 \quad \text{and} \\ \left| f_n\left(\frac{1}{1+n}\right) - f\left(\frac{1}{1+n}\right) \right| &= \left(\frac{n}{n+1}\right)^{n+1}. \end{aligned}$$

Now a is fixed, so as $n \to \infty$, we have $na(1-a)^n \to 0$. Also because a is fixed, with large enough values of n we will have $1/(1+n) < a$, so the third extremal point will not be on M. This means the convergence is uniform.

b) Second, let $M = [0, 1]$.

This is the same case, except the third extremal point, $x = 1/(n+1)$, will always be on M. Because

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1} = e^{-1} > 0,$$

we have

$$\sup\{|f_n(x) - f(x)| \mid x \in M\} = \left(\frac{n}{n+1}\right)^{n+1}$$

and

$$\sup\{|f_n(x) - f(x)| \mid x \in M\} \to e^{-1} \neq 0 \quad \text{as } n \to \infty,$$

so the convergence is not uniform.

* * *

Footnote: The tedious calculation bits.

1) Solving $n(1-x)^n - n^2x(1-x)^{n-1} = 0$ for x. We take the common factor $n(1-x)^{n-1}$, and get

$$n(1-x)^{n-1}(1-(1+n)x) = 0.$$

This holds when either $(1-x)^{n-1} = 0$ or $1-(1+n)x = 0$. The first gives $x = 1$, which is not interesting. The second gives $x = 1/(1+n)$.

The first zero has a multiplicity $n-1$ (*on* $(n-1)$*-kertainen nollakohta*) and the second is of multiplicity 1 (*on yksinkertainen nollakohta*); this sums to n zeros of a n:th-degree polynomial, so we can be sure we've gotten all zeroes there are.

2) When a is fixed, $na(1-a)^n \to 0$ as $n \to \infty$. To see this, notice that as $0 < a < 1$, we have $0 < (1-a) < 1$, and so $(1-a)^n \to 0$. That $nb^n \to 0$ when $n \to \infty$ for all b such that $|b| < 1$ is known. Also, remembered, possibly.

3) It is well known that $\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x$. Since $\frac{n}{n+1} = 1 + \frac{-1}{n+1}$, we have

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \lim_{k\to\infty}\left(1+\frac{-1}{k}\right)^k = e^{-1}.$$

(2) Prove: If $m^*(B) = 0$, then $m^*(A \cup B) = m^*(A)$.

* * *

By subadditivity

$$m^*(A\cup B) \leq m^*(A) + m^*(B) = m^*(A).$$

Since $A \cup B \supset A$, by monotonicity $m^*(A\cup B) \geq m^*(A)$. These combined give the result.

(3) Prove Corollary 2.4: If $A \subset \mathbb{R}^n$ is countable, then $m^*(A) = 0$.

* * *

Since A is countable, we can write

$$A = \{a_1, a_2, \ldots\} = \bigcup_{i=1}^{\infty} \{a_i\}.$$

Since m^* is subadditive, we have

$$0 \leq m^*(A) \leq \sum_{i=1}^{\infty} m^*(\{a_i\}) = \sum_{i=1}^{\infty} 0 = 0.$$

(4) Prove Theorem 2.6: Outer measure m^* is translation invariant, that is, if $a \in \mathbb{R}$, then $m^*(A + a) = m^*(A)$ for all $A \subset \mathbb{R}$.

* * *

In the lectures we defined the translation of a set E with $a \in \mathbb{R}$ as

$$a + E = \{a + x \mid x \in E\}.$$

We've defined the outer measure as

$$m^*(A) = \inf \left\{ \sum_n l(I_n) \mid I_n\text{'s are open} \right.$$
$$\left. \text{intervals so that} A \subset \bigcup_n I_n \right\}.$$

If I_n is such a sequence of open intervals that $A \subset \cup_n I_n$, then $A + a \subset \cup_n I_n + a$. This means any intervals that "work" for the definition of $m^*(A)$ have a-shifted intervals that "work" for the definition of $m^*(A+a)$. Because the infimum which determines $m^*(A+a)$ has at least elements as long as all those elements that determine the infimum of $m^*(A)$, we know that $m^*(A + a) \leq m^*(A)$.

But if I_n is such a sequence of intervals so that $A + a \subset \cup_n I_n$, then $A \subset \cup_n I_n - a$. Thus $m^*(A + a) \geq m^*(A)$.

(5) Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Prove that

$$\sum_n l(I_n) \geq 1.$$

* * *

Either $[0,1] \subset \cup_n I_n$ or not. If yes, then

$$\begin{aligned} 1 = l([0,1]) = m^*([0,1]) &\leq m^*(\cup_n I_n) \\ &\leq \sum_n m^*(I_n) = \sum_n l(I_n), \end{aligned}$$

and we are done. If not, then there are points $x \in [0,1]$ so that $x \notin \cup_n I_n$. Because these points x are not covered by any I_n, they are not rational points. Because we know there is a rational point between any two irrational points, we know these points x are all isolated by the intervals I_n. The only way an isolated irrational point can be not in the union of finite I_n is if it is a shared endpoint for two I_n: say $I_i =]a,x[$ and $I_j =]x,b[$. Because there is only a finite number of I_n, they have a finite number of endpoints, so there is only a finite number of points x. As they are a finite set they are less than countable, their length is zero (see Corollary 2.4); and so by Problem (2) above we have

$$\begin{aligned} m^*([0,1]) &= m^*((\cup_n I_n) \cap [0,1]) + \underbrace{m^*(\cup\{x\})}_{=0} \\ &= m^*((\cup_n I_n) \cap [0,1]). \end{aligned}$$

Now

$$\begin{aligned} 1 = m^*((\cup_n I_n) \cap [0,1]) &\leq m^*(\cup_n I_n) \\ &\leq \sum_n m^*(I_n) = \sum_n l(I_n). \end{aligned}$$