Analysis IV
Spring 2011
Exercises 5

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(1) Let Ω be a arbitrary set. Let f be a function from all subsets of Ω to $\mathbb{R}$, defined as

$$f(A) = \begin{cases} 0 & \text{if } A = \emptyset, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Show that f is monotone and subadditive, that is,
a) if $A \subset B \subset \Omega$ then $f(A) \leq f(B)$, and
b) for any $A_i \subset \Omega$ it holds that $f(\cup_i A_i) \leq \sum_i f(A_i)$.

(2) Let f be a function, and let us define $f^+(x) = \max\{0, f(x)\}$ and $f^-(x) = \max\{0, -f(x)\}$. Show that
a) $f^+(x) - f^-(x) = f(x)$,
b) $f^+(x) + f^-(x) = |f(x)|$ and
c) $\frac{1}{2}(|f| + f) = f^+$.

(3) Let A be a non-measurable set. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \notin A. \end{cases}$$

Is f measurable? How about $|f|$?

(4) In the lectures, we asserted that

$$\begin{aligned} &\{x \in E \mid f(x) \geq \alpha\} \\ =&\bigcap_{i=1}^{\infty} \{x \in E \mid f(x) > \alpha - 1/n\}. \end{aligned}$$

Prove it.

(5) In the lectures, we asserted that

$$\begin{aligned} &\{x \in E \mid (f+g)(x) < \alpha\} \\ =&\bigcup_{r \in \mathbb{Q}} \{x \in E \mid f(x) < r\} \cap \{x \in E \mid \alpha - g(x) > r\}. \end{aligned}$$

Prove it.