Analysis IV
Spring 2011
Exercises 5 / Answer

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(1) Let Ω be a arbitrary set. Let f be a function from all subsets of Ω to $\mathbb{R}$, defined as

$$f(A) = \begin{cases} 0 & \text{if } A = \emptyset, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Show that f is monotone and subadditive, that is,
a) if $A \subset B \subset \Omega$ then $f(A) \le f(B)$, and
b) for any $A_i \subset \Omega$ it holds that $f(\cup_i A_i) \le \sum_i f(A_i)$.

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Monotonicity: If $A \subset B$ then $f(A) \le f(B)$.
- If $B = \emptyset$ then $A = \emptyset$ and $f(A) = 0 = f(B)$.
- If $B \neq \emptyset$ then $f(B) = 1$, and $f(A) \le 1 = f(B)$ for all A.

Subadditivity: For any A_i it holds that $f(\cup_i A_i) \le \sum_i f(A_i)$.
- If $\cup_i A_i = \emptyset$, then $A_i = \emptyset$ for each i. As $0 \le 0$, the condition holds.
- If $\cup_i A_i \neq \emptyset$, the $A_i \neq \emptyset$ for at least one i, and so $f(\cup_i A_i) = 1 \le \sum_i f(A_i)$.

If this course was about more general measure theory and not just the measure theory of $\mathbb{R}$, we would define the general outer measure to be any function from sets to real numbers that was monotone, subadditive and mapped the empty set to zero — that is, the function f of this problem is a general outer measure.

(2) Let f be a function, and let us define $f^+(x) = \max\{0, f(x)\}$ and $f^-(x) = \max\{0, -f(x)\}$. Show that
a) $f^+(x) - f^-(x) = f(x)$,
b) $f^+(x) + f^-(x) = |f(x)|$ and
c) $\frac{1}{2}(|f| + f) = f^+$.

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Simple calculation:

$$f^+(x) - f^-(x) = \begin{cases} f(x) - 0 & \text{if } f(x) \ge 0 \\ 0 - (-f(x)) & \text{if } f(x) < 0 \end{cases} = f(x).$$

Also,

$$f^+(x) + f^-(x) = \begin{cases} f(x) + 0 & \text{if } f(x) \geq 0 \\ 0 + (-f(x)) & \text{if } f(x) < 0 \end{cases} = |f(x)|.$$

By adding the previous two results, we get

$$2f^+(x) - f^-(x) + f^-(x) = f(x) + |f(x)|,$$

which gives the third once we divide by two.

(3) Let A be a non-measurable set. Let

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ -1 \text{ if } x \notin A. \end{cases}$$

Is f measurable? How about $|f|$?

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The function f is measurable if $\{x \mid f(x) > \alpha\}$ is a measurable set for all $\alpha \in \mathbb{R}$. We choose $\alpha = 0$. Then

$$\{x \mid f(x) > 0\} = A,$$

and as this is not a measurable set, f is not a measurable function.

The absolute value of f is define as

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ |-1| \text{ if } x \notin A \end{cases} = 1,$$

and as $\{x \mid |f(x)| > \alpha\}$ is, depending on the value of α, always either $\emptyset$ or $\mathbb{R}$, which both are measurable sets, we know $|f|$ is a measurable function.

(4) In the lectures, we asserted that

$$\{x \in E \mid f(x) \geq \alpha\} = \bigcap_{i=1}^{\infty} \{x \in E \mid f(x) > \alpha - 1/n\}.$$

Prove it.

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This is the easiest to show by showing both sides are a subset of the other.

"$\subset$" Let $x \in E$ belong to the left-hand side set, that is, let $f(x) \geq \alpha$. Let $n \in \mathbb{N}$ be arbitrary (*mielivaltainen*). Then

$$f(x) \geq \alpha > \alpha - 1/n.$$

Thus $x \in \{x \in E \mid f(x) > \alpha - 1/n\}$ for every n, and so x is also in their intersection.

"$\supset$" Let $x \in E$ belong to the right-hand side set, that is, let $f(x) > \alpha - 1/n$ for each $n \in \mathbb{N}$. Does this mean that $f(x) \geq \alpha$? Let us assume for absolute clarity's sake (*rautalangasta vääntääksemme*) that $f(x) = \alpha - h$ instead, for some $h > 0$. In that case we can still choose n so that $n > 1/h$ and get

$$\alpha - 1/n > \alpha - h,$$

which is a contradiction. Thus $f(x) \geq \alpha$, and x is in the left-hand side set.

Because the left-hand side and the right-hand side sets are subsets of each other, they are exactly the same set.

(The proof of

$$\{x \in E \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) < \alpha + 1/n\}$$

is very similar.)

(5) In the lectures, we asserted that

$$\{x \in E \mid (f+g)(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \{x \in E \mid f(x) < r\} \cap \{x \in E \mid \alpha - g(x) > r\}.$$

Prove it.

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This is done the same way as the previous exercise.

"$\subset$" Let $x \in E$ so that $(f+g)(x) = f(x) + g(x) < \alpha$. Then $f(x) < \alpha - g(x)$. Now x is in the right-hand side set if there exists a rational number r so that $f(x) < r$ and $r < \alpha - g(x)$. As $f(x) \neq \alpha - g(x)$ they values $f(x)$ and $\alpha - g(x)$ are distinct real numbers, and we know we can always find some rational number r between them.

"$\supset$" Let $x \in E$ so that for some rational number r the following holds: $f(x) < r$ and $r < \alpha - g(x)$. Then clearly $f(x) < \alpha - g(x)$, or $f(x) + g(x) < \alpha$, so x is in the left-hand side set.