Analysis IV
Spring 2011
Exercises 6

(1) Show: If E_1 and E_2 are measurable, then $E_1 \cup E_2$ is measurable.

Hint. If E_2 is measurable, the condition of Definition 2.7 holds for $A \cap (\mathbb{R} \setminus E_1)$.

(2) Assuming (a, ∞) is measurable for all $a \in \mathbb{R}$, show that all other real intervals are measurable.

Hint. To show this, show it for intervals $(-\infty, b]$, $(-\infty, b)$, (a, b) and $[a, b]$. You will probably need the part of Theorem 2.12 which says countable intersections of measurable sets are measurable. This hasn't been proven but is sensible. Because

$$E_1 \cap E_2 = \mathbb{R} \setminus ((\mathbb{R} \setminus E_1) \cup (\mathbb{R} \setminus E_2)),$$

so the previous problem and Theorem 2.8 imply that $E_1 \cap E_2$ is measurable if E_1 and E_2 are measurable. Going from this to the measurability of countable unions and intersections would be a bit more complicated.

(3) Assuming $E_n \subset \mathbb{R}$, $n = 1, 2, \ldots$, are disjoint, show that

$$m(\bigcup_n E_n) = \sum_n m(E_n).$$

Hint. Use induction to show this for $E_1, \ldots, E_n$. Then notice that as clearly $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^\infty E_i$ holds for any finite n, the result can be extended for any $E_1, E_2, \ldots$.

(4) Let f and g be measurable functions. Show that $\max\{f, g\}$ is a measurable function.

(5) Let $f : E \to \hat{\mathbb{R}}$ be a measurable function. Show that $\{x \in E \mid f(x) = r\}$ is a measurable set.

Note:

$(a, b) =]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$	open interval / avoin väli
$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$	closed interval / suljettu väli
$(a, b] =]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$	semi-open interval / puoliavoin väli

It seems to be the case that (a, b) is the more common "American" way of denoting an open interval and $]a, b[$ the "European" way. Either one is fine.