Analysis IV
Spring 2011
Exercises 6 / Answers

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(1) Show: If E_1 and E_2 are measurable, $E_1 \cup E_2$ is measurable.

Hint. If E_2 is measurable, the condition of Definition 2.7 holds for $A \cap (\mathbb{R} \setminus E_1)$.

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We know that a set $E \subset \mathbb{R}$ is said to measurable, if for every other set $A \subset \mathbb{R}$ the following holds:[1]

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Because we want to show that $E_1 \cup E_2$ is measurable, we want to show that

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \setminus (E_1 \cup E_2)).$$

for all $A \subset \mathbb{R}$. As always, we know "$\leq$" holds (subadditivity), so we only have to show "$\geq$".

First, since E_2 is measurable, we have for the set $A \cap (\mathbb{R} \setminus E_1)$, where $A \subset \mathbb{R}$, the following:

$$\begin{aligned} & m^*(A \cap (\mathbb{R} \setminus E_1)) \\ = {} & m^*(A \cap (\mathbb{R} \setminus E_1) \cap E_2) \\ & + m^*(\underbrace{(A \cap (\mathbb{R} \setminus E_1)) \setminus E_2}_{= A \cap (\mathbb{R} \setminus E_1) \cap (\mathbb{R} \setminus E_2)}). \end{aligned}$$

Second, since $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap (\mathbb{R} \setminus E_1))$, we have that

$$\begin{aligned} & m^*(A \cap (E_1 \cup E_2)) \\ & \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap (\mathbb{R} \setminus E_1)). \end{aligned}$$

[1] Some books write this as $m^*(A) = m^*(A \cap E) + m^*(A \cap (\mathbb{R} \setminus E))$, which is the same thing written in a different way. Whichever way seems more elegant to you is fine.

Third, since E_1 is measurable,

$$\begin{aligned} &m^*(A \cap (E_1 \cup E_2)) \\ &\quad + m^*(A \cap (\mathbb{R} \setminus E_1) \cap (\mathbb{R} \setminus E_2)) \\ &\le m^*(A \cap E_1) + m^*(A \cap E_2 \cap (\mathbb{R} \setminus E_1)) \\ &\quad + m^*(A \cap (\mathbb{R} \setminus E_1) \cap (\mathbb{R} \setminus E_2)) \\ &= m^*(A \cap E_1) + m^*(A \cap (\mathbb{R} \setminus E_1) = m^*(A). \end{aligned}$$

Note. This could be generalized to show $\bigcup_{i=1}^n E_i$ is measurable. The union of countable measurable sets is also measurable, but this would be more difficult to show.

(2) Assuming (a, ∞) is measurable for all $a \in \mathbb{R}$, show that all other real intervals are measurable.

Hint. Because $E_1 \cap E_2 = \mathbb{R} \setminus ((\mathbb{R} \setminus E_1) \cup (\mathbb{R} \setminus E_2))$, the previous problem and Theorem 2.8 imply that $E_1 \cap E_2$ is measurable.

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It is shown on p. 59 of Royden's book that (a, ∞) is measurable for all $a \in \mathbb{R}$.

(a) Since every interval (a, ∞) is measurable, their complements, the intervals $(-\infty, a]$, are measurable.

(b) Since every interval $(-\infty, a]$ is measurable, their countable intersections

$$\bigcap_{n=1}^{\infty} (-\infty, a + 1/n] = (-\infty, a)$$

are measurable.

(c) Since every interval $(-\infty, a)$ is measurable, their complements, the intervals $[a, \infty)$, are measurable.

(d) Since $(-\infty, b)$ and (a, ∞) are measurable for any $a, b \in \mathbb{R}$, we know

$$(a, b) = (-\infty, b) \cap (a, \infty)$$

is measurable. Here $a < b$.

(e) Similarly, the intervals

$$[b, a] = (-\infty, b] \cap [a, \infty)$$

are measurable.

(3) Assuming $E_n \subset \mathbb{R}$, $n = 1, 2, \ldots$, are disjoint, show that

$$m(\bigcup_n E_n) = \sum_n m(E_n).$$

Hint. Clearly $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^\infty E_i$ holds for any finite n.

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1. Proof for $n = 1, 2, \ldots, N$. By induction. If $N = 1$, clearly

$$m(\bigcup_{n=1}^{1} E_n) = m(E_1) = \sum_{n=1}^{1} m(E_n).$$

Assume the statement holds for $n = N - 1$. In the case $n = N$, since the sets E_n are disjoint, we have

$$\bigcup_{n=1}^{N} E_n \cap E_N = E_N$$

and

$$\bigcup_{n=1}^{N} E_n \setminus E_N = \bigcup_{n=1}^{N-1} E_n.$$

Because E_N is measurable,

$$\begin{aligned} m^*(\bigcup_{n=1}^{N} E_n) &= m^*(\bigcup_{n=1}^{N} E_n \cap E_N) + m^*(\bigcup_{n=1}^{N} E_n \setminus E_N) \\ &= m^*(E_N) + m^*(\bigcup_{n=1}^{N-1} E_n) \\ &= m^*(E_N) + \sum_{n=1}^{N-1} m^*(E_n). \end{aligned}$$

2. For any finite n we have

$$\sum_{i=1}^{n} m(E_i) = m(\bigcup_{i=1}^{n} E_i) \leq m(\bigcup_{i=1}^{\infty} E_i).$$

As the right-hand side does not depend on n,

$$\sum_{i=1}^{\infty} m(E_i) \leq m(\bigcup_{i=1}^{\infty} E_i).$$

The opposite inequality comes from subadditivity.

(4) Let f and g be measurable functions. Show that $\max\{f, g\}$ is a measurable function.

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Because f and g are measurable, $f - g$ is measurable (see Theorem 2.17). Because $f - g$ is measurable, $(f-g)^+$ is measurable. Because g and $(f-g)^+$ are measurable, $g + (f-g)^+$ is measurable.

Since

$$g + (f-g)^+ = g + \max\{f-g, 0\} = \begin{cases} g + f - g & \text{if } f \geq g \\ g + 0 & \text{if } f < g \end{cases} = \max\{f, g\},$$

$\max\{f, g\}$ is measurable.

(5) Let $f : E \to \hat{\mathbb{R}}$ be a measurable function. Show that $\{x \in E \mid f(x) = r\}$ is a measurable set.

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Sets $\{x \mid f(x) \geq r\}$ and $\{x \mid f(x) \leq r\}$ are measurable. Their intersection, too.

11. Lemma: *The interval (a, ∞) is measurable.*

Proof: Let A be any set, $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$. Then we must show $m^*A_1 + m^*A_2 \leq m^*A$. If $m^*A = \infty$, then there is nothing to prove. If $m^*A < \infty$, then, given $\epsilon > 0$, there is a countable collection $\{I_n\}$ of open intervals which cover A and for which

$$\sum l(I_n) \leq m^*A + \epsilon.$$

Let $I_n' = I_n \cap (a, \infty)$ and $I_n'' = I_n \cap (-\infty, a]$. Then I_n' and I_n'' are intervals (or empty) and

$$l(I_n) = l(I_n') + l(I_n'') = m^*I_n' + m^*I_n''.$$

Since $A_1 \subset \bigcup I_n'$, we have

$$m^*A_1 \leq m^*\left(\bigcup I_n'\right) \leq \sum m^*I_n',$$

and, since $A_2 \subset \bigcup I_n''$, we have

$$m^*A_2 \leq m^*\left(\bigcup I_n''\right) \leq \sum m^*I_n''.$$

Thus

$$\begin{aligned} m^*A_1 + m^*A_2 &\leq \sum (m^*I_n' + m^*I_n'') \\ &\leq \sum l(I_n) \leq m^*A + \epsilon. \end{aligned}$$

But ϵ was an arbitrary positive number, and so we must have $m^*A_1 + m^*A_2 \leq m^*A$. ∎