Analysis IV
Spring 2011
Exercises 7 / Answers

(1) Show that if f is a measurable function and $[a,b]$ is a real interval, the set

$$\{x \in \mathbb{R} \mid f(x) \in [a,b]\}$$

is measurable.

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Because $x \in [a,b]$ if and only if $x \geq a$ and $x \leq b$, we have

$$\begin{aligned}&\{x \in \mathbb{R} \mid f(x) \in [a,b]\}\\ &= \{x \in \mathbb{R} \mid f(x) \geq a\} \cap \{x \in \mathbb{R} \mid f(x) \leq b\}.\end{aligned}$$

(2) Prove Theorem 2.26 (b'): If $f \geq g$ and $\int f\,dm$ exists and $\int f\,dm < \infty$, then $\int g\,dm$ exists and

$$\int f\,dm \geq \int g\,dm.$$

* * *

The proof is similar to that of (b).

We may as well suppose that $f \geq g \geq 0$. Let ϕ be a simple function so that $0 \leq \phi \leq g$. Then $0 \leq \phi \leq f$, and by Definition 2.24 we have

$$\int \phi\,dm \leq \int f\,dm.$$

Since this holds for any simple function ϕ with $\phi \leq g$, we have

$$\int g\,dm = \sup\{\int \phi\,dm \mid \phi \text{ simple and } \phi \leq g\} \leq \int f\,dm$$

for all functions f and g. Because $f \geq g$, we have $f^+ \geq g^+$ and $f^- \leq g^-$. (See note.[1]) Further,

$$\int f^+\,dm - \int f^-\,dm$$

[1]This is obvious if you draw a picture, but a little bit messy to show. Remember that $f(x) = f^+(x) - f^-(x)$, and similarly for g. At any specific point x, we have either $f(x) = f^+(x)$ (if $f(x) \geq 0$) or $f(x) = -f^-(x)$ (if $f(x) < 0$). Let us consider the case $f(x) = f^+(x)$. At that point x either $g(x) = g^+(x)$ or $g(x) = -g^-(x)$. In the first case, since we have $f \geq g$ everywhere, we have $f^+(x) \geq g^+(x)$. In the second case, we have $g(x) < 0$, so $g^+(x) = 0$, and $f^+(x) \geq 0$. The other inequality comes from the case $f(x) = -f^-(x)$.

is defined, since $\int f\,dm$ exists. Since $\int f\,dm < \infty$, we know $\int f^+\,dm < \infty$, and thus

$$\int g^+\,dm \leq \int f^+\,dm < \infty.$$

Therefore $\int g\,dm$ exists, and we have

$$\int g\,dm = \int g^+\,dm - \int g^-\,dm \leq \int f^+\,dm - \int f^-\,dm = \int f\,dm.$$

(3) Let f be a non-negative measurable function. Show that $\int f\,dm = 0$ implies $f = 0$ a.e..

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Assume the opposite: $\int f\,dm = 0$ and $f > 0$ in some $E \subset \mathbb{R}$, $m(E) > 0$.

We can assume $f > a > 0$ for some $a \in \mathbb{R}$ on $A \subset E$, $m(A) > 0$.[2] This means $f \geq \phi$ for the simple function

$$\phi(x) = \begin{cases} a, & x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Because

$$\int \phi\,dm = a\,m(A) > 0,$$

by Theorem 2.26(d) we have

$$\int f\,dm = \sup\left\{\int \phi\,dm \mid \phi \text{ simple and } \phi \leq f\right\} \geq a\,m(A) > 0,$$

which is a contradiction.

(4) Let f be a measurable function. Show that if E is a measurable set and $m(E) = 0$, then $\int_E f\,dm = 0$.

Hint. Prove this first for simple functions. Then for $f \geq 0$ using Theorem 2.26(d). Then note that $f = f^+ - f^-$.

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[2] If for all $a > 0$ the set where $f > a$ was of zero measure, we could say

$$E = \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > 1/n\},$$

and have

$$m(E) \leq \sum_{n=1}^{\infty} m(\{x \in E \mid f(x) > 1/n\}) = 0,$$

which would be a contradiction because $m(E) > 0$.

1. Let $m(E) = 0$ and let f be a simple function, that is, let

$$f(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x)$$

for some $A_j \subset \mathbb{R}$, $a_j \in \mathbb{R}$, $j = 1, \ldots, n$. By definition, $\int_E f\,dm$ is

$$\int_E f\,dm = \int f \chi_E\,dm = \sum_{j=1}^{n} a_j m(A_j \cap E).$$

Since $m(E) = 0$ and $A_j \cap E \subset E$ for each j, we have $m(A_j \cap E) = 0$ for each j, and so

$$\sum_{j=1}^{n} a_j m(A_j \cap E) = 0.$$

2. Let $m(E) = 0$ and let f be a non-negative function. By Theorem 2.26(d),

$$\int_E f\,dm = \sup\left\{\int_E \phi\,dm \mid \phi \text{ simple and } \phi \leq f\right\}.$$

Because $\int_E \phi\,dm = 0$ for all simple functions, $\int_E f\,dm = 0$.

3. Finally, let f be any measurable function. We can write $f = f^+ - f^-$, and as $f^+ \geq 0$ and $f^- \geq 0$, we know that

$$\int_E f^+\,dm = 0 \qquad \text{and} \qquad \int_E f^-\,dm = 0.$$

Since

$$\int_E f\,dm = \int_E f^+\,dm - \int_E f^-\,dm,$$

this means $\int_E f\,dm = 0$.

(5) Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 0, & x \in \mathbb{Q} \cap [0,1]. \end{cases}$$

Calculate

$$\int_{[0,1]} f\,dm.$$

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Since f only gets the values 0 and 1, the integral is by definition

$$\int_{[0,1]} f\,dm = 1 \cdot m((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) + 0 \cdot m(\mathbb{Q} \cap [0,1]).$$

We know that $\mathbb{Q}$ is countable, so its subset $\mathbb{Q} \cap [0,1]$ is also countable. By Corollary 2.4, $m(\mathbb{Q} \cap [0,1]) = 0$. Because

$$[0,1] = ((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) \cup (\mathbb{Q} \cap [0,1]),$$

we have by Ex. 4.2 that

$$m((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) = m([0,1]) = 1.$$

This means

$$\int_{[0,1]} f \, dm = 1 \cdot 1 + 0 \cdot 0 = 1.$$

(Note that the only fact we needed to remember of (ir)rational numbers is that the rational numbers are countable. All else was measure-theoretical trickery.)

(6) Prove Lemma 3.1.: If $a, b \geq 0$ and $0 < \lambda < 1$, then

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b.$$

Hint. Consider the cases $b = 0$ and $b \neq 0$ separately. Notice that $g : [0, \infty[\to \mathbb{R}$,

$$g(t) = (1-\lambda) + \lambda t - t^\lambda, \qquad 0 < \lambda < 1,$$

has its minimum at $t = 1$.

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If $b = 0$, the claim is that

$$0 \leq \lambda a$$

for all $a \geq 0$, $0 < \lambda < 1$. This is obviously true.

If $b \neq 0$, we can divide the claim by b, and get

$$a^\lambda b^{-\lambda} \leq \lambda a / b + (1-\lambda).$$

If we write a/b as t, this is

$$t^\lambda \leq \lambda t + (1-\lambda)$$

or

$$0 \leq \lambda t + (1-\lambda) - t^\lambda.$$

For $g(t) = \lambda t + (1-\lambda) - t^\lambda$, we have $g(1) = \lambda + (1-\lambda) - 1 = 0$. By the hint this is the minimum of g, and thus the inequality

$$0 \leq \lambda t + (1-\lambda) - t^\lambda$$

holds for all $t \in [0, \infty[$, $0 < \lambda < 1$.