Analysis IV
Spring 2011
Exercises 8 ($\Leftrightarrow$ First exam)

(1) Let $X \neq \emptyset$. Define a function $d : X \times X \to \mathbb{R}$ by the equality

$$d(x,y) = \begin{cases} 1, & \text{as } x \neq y \\ 0, & \text{as } x = y. \end{cases}$$

Show that d is a metric in X.

(2) Let M be a non-empty set in the space $\mathbb{R}^n$, and let

$$M_r = \{x \in \mathbb{R}^n \mid d(x, M) < r\}$$

for all $r > 0$. Prove that M_r is an open set in $\mathbb{R}^n$ for all $r > 0$. (Note that $d(x, M) = \inf\{d(x,y) \mid y \in M\}$.)

(3) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric space (M, d). Prove that there exists $R > 0$ such that $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

(4) Show that if E_1 and E_2 are measurable sets then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

(5) Let A be a non-measurable set, and let

$$f(x) = \begin{cases} 1, & \text{if } x \in A \text{ and} \\ -1, & \text{if } x \notin A. \end{cases}$$

Is f measurable? How about $|f|$?