Analysis IV
Spring 2011
Exercises 8 ($\Leftrightarrow$ First exam) / Answers

(1) Let $X \neq \emptyset$. Define a function $d : X \times X \to \mathbb{R}$ by the equality

$$d(x, y) = \begin{cases} 1, & \text{as } x \neq y \\ 0, & \text{as } x = y. \end{cases}$$

Show that d is a metric in X.

$* * *$

The function d is a metric if the following four conditions are true:

(a) $d(x, y) \geq 0$
Because $d(x, y) \in \{0, 1\}$ for all x, y, $d(x, y) \geq 0$.
(b) $d(x, y) = 0 \Leftrightarrow x = y$
Clear by the definition of d.
(c) $d(x, y) = d(y, x)$
Since the equality $=$ is commutative, d is commutative.
(d) $d(x, z) \leq d(x, y) + d(y, z)$
The left-hand side can take the values 0 and 1. The right-hand side can take the values 0, 1 and 2. The only case where the inequality would not hold would be when the left-hand side was 1 and the right-hand side 0. If the left-hand side is 1, then $x \neq z$. But then either $x \neq y$ or $y \neq z$, as else $x = y = z$; and hence the right-hand side is 1 or more.

or

- If $x = y = z$, both sides are 0; OK.
- If $x \neq y = z$ or $x = y \neq z$, the inequality is $0 \leq 1$ ($x = z$) or $1 \leq 1$ ($x \neq z$), which is OK.
- If $x \neq y \neq z$, we have either $0 \leq 2$ ($x = z$) or $1 \leq 2$ ($x \neq z$); both are OK.

(2) Let M be a non-empty set in the space $\mathbb{R}^n$, and let

$$M_r = \{x \in \mathbb{R}^n \mid d(x, M) < r\}$$

for all $r > 0$. Prove that M_r is an open set in $\mathbb{R}^n$ for all $r > 0$. (Note that $d(x, M) = \inf\{d(x, y) \mid y \in M\}$.)

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Let $x \in M_r$. (Note that we don't assume $x \in M$; and we don't assume or prove that M is open.) We will show there is an environment, more exactly an open ball B, so that $x \in B \subset M_r$. This shows that M_r is open.

Since $x \in M_r$, we know that $d(x, M) < r$. Because this is a strict inequality,[1] we have for some $\epsilon > 0$ that $d(x, M) = r - \epsilon$.

If we take the ball $B(x, \epsilon)$, we have $B(x, \epsilon) \subset M_r$. To see this, take a point $x_1 \in B(x, \epsilon)$, that is, a point x_1 with $d(x, x_1) < \epsilon$. Then

$$\begin{aligned} d(x_1, M) &= \inf\{d(x_1, y) \mid y \in M\} \\ &\leq \inf\{d(x_1, x) + d(x, y) \mid y \in M\} \\ &= d(x, x_1) + d(x, M) \\ &< \epsilon + r - \epsilon \\ &= r. \end{aligned}$$

(3) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric space (M, d). Prove that there exists $R > 0$ such that $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

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See Exercises 2 problem 4.

(4) Show that if E_1 and E_2 are measurable sets then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

$* * *$

We can write

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2).$$

As these three sets are disjoint, we have

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2).$$

[1] Just in case: both "$<$" and "$\leq$" denote *inequalities*; and we call "$<$" the *strict* inequality: "strictly less than" (*"aidosti vähemmän kuin"*).

We can now add $m(E_1 \cap E_2)$ to both sides and write

$$\begin{aligned} &m(E_1 \cup E_2) + m(E_1 \cap E_2) \\ =&\underbrace{m(E_1 \setminus E_2) + m(E_1 \cap E_2)}_{=m(E_1) \text{ (disjoint sets)}} + \underbrace{m(E_2 \setminus E_1) + m(E_1 \cap E_2)}_{=m(E_2) \text{ (disjoint sets)}}. \end{aligned}$$

(5) Let A be a non-measurable set, and let

$$f(x) = \begin{cases} 1, & \text{if } x \in A \text{ and} \\ -1, & \text{if } x \notin A. \end{cases}$$

Is f measurable? How about $|f|$?

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See Exercises 5 problem 3.