Analysis IV
Spring 2011
Exercises 9 / Answers

(1) Prove Lemma 3.9: If $\{f_n\}$ is a Cauchy sequence in the metric d_{L^p}, $1 \le p < \infty$, then $\{f_n\}$ is a Cauchy sequence in the measure m.

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Because the sequence $\{f_n\}$ is a Cauchy sequence in the metric d_{L^p}, we have for any $\epsilon > 0$ a number N so that

$$d(f_m, f_n) = \left(\int |f_n - f_m|^p \, dm \right)^{1/p} < \epsilon$$

for all $m, n > N$.

We say that $\{f_n\}$ is a Cauchy sequence in the measure m, if for each $\epsilon > 0$ and each $\delta > 0$ there exists a N such that

$$m(\{x \mid |f_n(x) - f_m(x)| \ge \epsilon\}) < \delta$$

for all $m, n > N$.

Suppose that $\{f_n\}$ is not a Cauchy sequence in m. Then there exists a pair ϵ_0 and δ_0 so that for any N we can find a pair $m, n > N$ so that

$$m(\{x \mid |f_n(x) - f_m(x)| \ge \epsilon_0\}) \ge \delta_0.$$

If we write

$$E = \{x \mid |f_n(x) - f_m(x)| \ge \epsilon_0\},$$

we have $m(E) \ge \delta_0$, and we get the following contradiction:

$$\left(\int |f_n - f_m|^p \, dm \right)^{1/p} \ge \left(\int_E |f_n - f_m|^p \, dm \right)^{1/p}$$
$$\ge \left(\int_E \epsilon_0^p \, dm \right)^{1/p} \ge \epsilon_0 \underbrace{\left(\int_E 1 \, dm \right)}_{= m(E)}^{1/p} \ge \epsilon_0 \delta_0^{1/p}.$$

(2) Prove Hölder's inequality for series: Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{a_n\} \in \ell^p$ and $\{b_n\} \in \ell^q$. Then $\{a_n b_n\} \in \ell^1$ and

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}.$$

* * *

If $\{a_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$ or $\{b_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$, both sides of the inequality are zero and we are done. Thus we can assume that $\sum_{n=1}^{\infty} |a_n|^p > 0$ and $\sum_{n=1}^{\infty} |b_n|^q > 0$.

We write $A = (\sum_{n=1}^{\infty} |a_n|^p)^{1/p}$ and $B = (\sum_{n=1}^{\infty} |b_n|^q)^{1/q}$. Note that A and B are constants that do not depend on n. By Young's inequality (Lemma 3.1)[1], we have

$$\frac{a_n}{A} \frac{b_n}{B} \leq \frac{a_n^p}{A^p p} + \frac{b_n^q}{B^q q}.$$

Summing over n, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{|a_n|}{A} \frac{|b_n|}{B} \\
&\leq \sum_{n=1}^{\infty} \left(\frac{|a_n|^p}{A^p p} + \frac{|b_n|^q}{B^q q} \right) \\
&= \frac{\sum_{n=1}^{\infty} |a_n|^p}{A^p p} + \frac{\sum_{n=1}^{\infty} |b_n|^q}{B^q q} \\
&= \frac{A^p}{A^p p} + \frac{B^q}{B^q q} = \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

When we multiply this with $AB = (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} (\sum_{n=1}^{\infty} |b_n|^q)^{1/q}$, we get

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}.$$

[1] Perhaps it's not immediately obvious how this is done. We choose $\lambda = 1/p$, and so get $1 - \lambda = 1 - 1/p = 1/q$ by the choice of p and q, and choose $a = (a_n/A)^p$ and $b = (b_n/B)^q$, and Lemma 3.1 simplifies into the form used.

(3) Prove Minkowski's inequality for series: Let $1 \leq p < \infty$. Suppose that $\{a_n\} \in \ell^p$ and $\{b_n\} \in \ell^p$. Then $\{a_n + b_n\} \in \ell^p$ and

$$\left(\sum_{n=1}^{\infty} |a_n + b_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p}.$$

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Because $|a_n + b_n| \leq 2^{p-1}|a_b| + 2^{p-1}|b_n|$ (see lectures), we know the left-hand side is well defined. If $\{a_n\} = \{0\}$, $\{b_n\} = \{0\}$ or $\{a_n + b_n\} = \{0\}$ the claim holds, so we can assume none of these three is a zero sequence. Next, we rearrange the sums:

$$\begin{aligned}
&\sum_{n=1}^{\infty} |a_n + b_n|^p = \sum_{n=1}^{\infty} |a_n + b_n||a_n + b_n|^{p-1} \\
&\leq \sum_{n=1}^{\infty} (|a_n| + |b_n|)\, |a_n + b_n|^{p-1} \\
&\leq \sum_{n=1}^{\infty} \left(|a_n||a_n + b_n|^{p-1} + |b_n||a_n + b_n|^{p-1}\right) \\
&\leq \sum_{n=1}^{\infty} |a_n||a_n + b_n|^{p-1} + \sum_{n=1}^{\infty} |b_n||a_n + b_n|^{p-1}.
\end{aligned}$$

Here we use Hölder's inequality. Note that the exponent q for which $\frac{1}{p} + \frac{1}{q} = 1$ holds is $q = p/(p-1)$.

$$\begin{aligned}
&\leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |a_n + b_n|^{(p-1)p/(p-1)}\right)^{(p-1)/p} \\
&\quad + \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |a_n + b_n|^{(p-1)p/(p-1)}\right)^{(p-1)/p} \\
&= \left(\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p}\right) \left(\sum_{n=1}^{\infty} |a_n + b_n|^p\right)^{1-1/p}.
\end{aligned}$$

We divide $\left(\sum_{n=1}^{\infty} |a_n + b_n|^p\right)^{1-1/p}$ to the left-hand side, and get

$$\left(\sum_{n=1}^{\infty} |a_n + b_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p}.$$

(4) Prove Lemma 3.15 for $p = \infty$: If $\{x_{n,k}\} \subset \ell^\infty$ is a Cauchy sequence, then there exists a uniform constant $C > 0$ such that

$$\sup_{k \in \mathbb{N}} |x_{n,k}| \le C$$

for all $n \in \mathbb{N}$.[2]

* * *

Since $\{x_{n,k}\}$ is a Cauchy sequence, we know that for each $\epsilon > 0$ there is a N so that

$$\sup_k |x_{m,k} - x_{n,k}| < \epsilon$$

when $m, n > N$. Let us choose $\epsilon = 1$. Then for all $n > N$ we have

$$\sup_k |x_{N+1,k} - x_{n,k}| < 1,$$

or

$$\sup_k |x_{n,k}| < 1 + \sup_k |x_{N+1,k}|.$$

Since each individual sequence $\{x_{n,k}\}_{k=1}^\infty \in \ell^\infty$, we have

$$\sup_k |x_{n,k}| < \infty$$

for each $n = 1, 2, \ldots, N$. Combining these two estimates, we have

$$\sup_k |x_{n,k}| < \underbrace{1 + \sup_k |x_{N+1,k}| + \max_{n=1,\ldots,N} \{\sup_k |x_{n,k}|\}}_{= C}$$

for all $n = 1, 2, \ldots$.

(Compare Ex. 2.4, which is very much like this problem.)

[2]Recall that $\{x_{n,k}\} \subset \ell^\infty$ means $\{x_{n,k}\}$ is a sequence of sequences (*jonojen jono*) with each element (with a fixed n) being a sequence $x_{n,k} \in \ell^\infty$. (And $\{x_{n,k}\} \in \ell^\infty$ is defined as $\sup_k |x_{n,k}| < \infty$.) The index n tells which sequence we are dealing with; the index k which element of that n:th sequence. That is,

$$\{x_{n,k}\} = \{\{x_{1,k}\}, \{x_{2,k}\}, \ldots\} = \{\{x_{1,k}\}_{k=1}^\infty, \{x_{2,k}\}_{k=1}^\infty, \ldots\}.$$

(To be really consistent, we should write $\{\{x_{n,k}\}\}$ instead of $\{x_{n,k}\}$, that is, a sequence (the first $\{\ldots\}$) of sequences $\{x_{n,k}\}$; but that would not look good.)

(5) Let $1 \leq p < q < \infty$. Define f and g as $f : [0, 2\pi] \to \hat{\mathbb{R}}$ and $g : [0, 2\pi] \to \hat{\mathbb{R}}$,

$$f(\theta) = \theta^{-1/q} \qquad \text{and} \qquad g(\theta) = \theta^{-1/2q}.$$

Show that $f \in L^p[0, 2\pi]$, $f \notin L^q[0, 2\pi]$, $g \in L^q[0, 2\pi]$, and $g \notin L^\infty[0, 2\pi]$. (You can assume f and g to be measurable.)

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We remember that for $1 < p < \infty$, $f \in L^p(U)$ if $\int_U |f|^p \, dm < \infty$. Let us first consider the claim that $f \in L^p[0, 2\pi]$. Because $p < q$, we know that $-p/q > -1$, and $1 - p/q > 0$, so

$$\int |f|^p \, dm = \int_{[0,2\pi]} \theta^{-p/q} \, dm \stackrel{(1)}{=} \int_0^{2\pi} (-p/q)\theta^{1-p/q} \, d\theta < \infty.$$

Read that carefully. The equality (1) is just substituting the old Riemannian integral for our new Lebesgue integral; whenever we can calculate the Riemannian integral for something, the Lebesgue integral has the same value.

Similarly, for $f \notin L^q[0, 2\pi]$ we see that

$$\int |f|^q \, dm = \int_{[0,2\pi]} \theta^{-q/q} \, dm = \int_0^{2\pi} \theta^{-1} \, d\theta = \Big|_0^{2\pi} \ln \theta = \infty.$$

For $g \in L^q[0, 2\pi]$, we see that

$$\int |g|^q \, dm = \int_{[0,2\pi]} \theta^{-q/2q} \, dm = \int_0^{2\pi} \theta^{-1/2} \, d\theta = \Big|_0^{2\pi} \theta^{1/2}/2 = \sqrt{2\pi}/2 < \infty.$$

Finally, to see that $g \notin L^\infty[0, 2\pi]$ we have to see that the condition

$$\operatorname{ess\,sup}_{[0,2\pi]} |g| < \infty$$

does not hold. But since $\sup_{[0,2\pi]} |g| = \infty$ (when $\theta \to 0$), and g is a continuous function, the supremum of g is infinity even outside any arbitrary set of zero measure, so

$$\operatorname{ess\,sup}_{[0,2\pi]} |g| = \infty.$$

(6) Let $f \in L^1$ and $g \in L^\infty$. Show that

$$\int |fg|\, dm \leq d_{L^1}(f,0)\, d_{L^\infty}(g,0).$$

* * *

Because $g \in L^\infty$, we know that

$$d_{L^\infty}(g,0) = \operatorname{ess\,sup} |g(x)| < \infty.$$

The value of an integral isn't affected by the values of a function in a set of measure zero. Thus

$$\int |fg|\, dm \leq \int |f| \underbrace{\operatorname{ess\,sup} |g|}_{\text{constant}}\, dm = d_{L^\infty}(g,0) \int |f|\, dm = d_{L^1}(f,0) d_{L^\infty}(g,0).$$

(If we wanted to be really explicit, we could note that the condition $\operatorname{ess\,sup} |g(x)| < \infty$ means $\sup_{X \setminus A} |g| < \infty$ for some set A with $m(A) = 0$. Then we'd note that

$$\int |fg|\, dm = \int_{X \setminus A} |fg|\, dm + \underbrace{\int_A |fg|\, dm}_{=0},$$

and then treat $\int_{X \setminus A} |fg|\, dm$ as we treat $\int |fg|\, dm$ above.)