Analysis IV
Spring 2011
Exercises 10

(1) Let $f_n : \mathbb{R} \to \mathbb{R}$, be defined as

$$f_n(x) = \begin{cases} 1/n^2, & \text{when } x \in [-n, n] \\ 0, & \text{elsewhere.} \end{cases}$$

Does f_n converge to $f(x) = 0$

(a) pointwise,
(b) in the measure m,
(c) with respect to d_{L^p} metric, $1 < p < \infty$,
(d) with respect to d_{L^∞} metric?

(2) Let f be defined as $f : [0, 1] \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. Calculate the norm of f in

(a) $C_{\mathbb{R}}([0, 1])$ and
(b) $L^1([0, 1])$.

(3) Show that *the standard norms of* ℓ^p are norms; that is, show that

$$\|\{x_n\}\|_p = \left(\sum_n |x_n|^p \right)^{1/p}$$

is a norm for $\{x_n\} \in \ell^p$, $1 < p < \infty$, and show that

$$\|\{x_n\}\|_\infty = \sup_n |x_n|$$

is a norm for $\{x_n\} \in \ell^\infty$.

(4) Show that in the space $C_{\mathbb{R}}([0, 1])$ the norms

$$\|f\|_1 = \int_0^1 (1 - t)|f(t)|\, dt$$

and

$$\|f\|_2 = \int_0^1 (1 - t^3)|f(t)|\, dt$$

are equivalent. (See Definition 4.4. In this and the next problem, you don't need to prove these functions are norms.)

(5) Let $P([0, 1])$ be the space of polynomials defined on $[0, 1]$. Show that the norms

$$\|p\|_a = \sup\{|p(x)| \mid x \in [0, 1]\}$$

and

$$\|p\|_b = \int_0^1 |p(x)|\, dx,$$

where $p \in P([0, 1])$, are not equivalent.