Analysis IV
Spring 2011
Exercises 10 / Answers

(1) Let $f_n : \mathbb{R} \to \mathbb{R}$, be defined as

$$f_n(x) = \begin{cases} 1/n^2, & \text{when } x \in [-n, n] \\ 0, & \text{elsewhere.} \end{cases}$$

Does f_n converge to $f(x) = 0$

(a) pointwise,
(b) in the measure m,
(c) with respect to d_{L^p} metric, $1 < p < \infty$,
(d) with respect to d_{L^∞} metric?

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The answers are yes, yes, yes and yes.

(a) pointwise convergence; that is, is $\lim_{n\to\infty} f_n(x) = f(x)$ for all x? First, note that with big enough n, any fixed point x is in $[-n, n]$. Thus we can assume $f_n(x) = 1/n^2$. Next, obviously

$$\lim_{n\to\infty} 1/n^2 = 0 = f(x),$$

so $f_n \to f$ pointwise.

(b) convergence in the measure m, that is, is there a N for each $\epsilon, \delta > 0$ so that $m(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) < \delta$ when $n \geq N$? The answer is yes, for for any $\epsilon > 0$ we can choose $N = 1 + \frac{1}{\sqrt{\epsilon}}$. Then $1/n^2 < \epsilon$ for every $n \geq N$, and as $|f_n(x) - f(x)| \leq 1/n^2$, we have

$$m(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) = 0 < \delta$$

regardless of the choice of δ.

(c) convergence with respect to d_{L^p} metric, $1 < p < \infty$, that is, is there a N so that

$$\left(\int |f_n - f|^p \, dm\right)^{1/p} < \epsilon$$

for all $n > N$? Let us see what is the value of that integral:

$$\left(\int |f_n - f|^p \, dm\right)^{1/p}$$
$$= \left(\int_{[-n,n]} 1/n^{2p} \, dm\right)^{1/p}$$
$$= \left(\frac{2n}{n^{2p}}\right)^{1/p} = 2^{1/p} n^{\frac{1}{p}-2}.$$

Since $1 < p < \infty$, we know $\frac{1}{p} - 2 < 0$. Then

$$\lim_{n\to\infty} n^{\frac{1}{p}-2} = 0,$$

and we have convergence in the d_{L^p} metric.

(d) convergence with respect to d_{L^∞} metric, that is, is there a N so that

$$\operatorname{ess\,sup} |f_n - f| < \epsilon$$

for all $n > N$? Now this is

$$\operatorname{ess\,sup} |f_n - f| = 1/n^2 < \epsilon;$$

are this will be true when $n > 1/\sqrt{\epsilon}$.

(2) Let f be defined as $f : [0,1] \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. Calculate the norm of f in

(a) $C_{\mathbb{R}}([0,1])$ and
(b) $L^1([0,1])$.

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(a) The $C_{\mathbb{R}}([0,1])$-norm of f is

$$\|f\| = \sup\{f(x) \mid x \in [0,1]\} = \sup\{x^n \mid x \in [0,1]\} = 1.$$

(b) The $L^1([0,1])$-norm of f is

$$\|f\| = \int_{[0,1]} x^n \, dm = \Big/_{0}^{1} \frac{1}{n+1} x^{n+1} = \frac{1}{n+1}.$$

(3) Show that *the standard norms of* ℓ^p are norms; that is, show that

$$\|\{x_n\}\|_p = \left(\sum_n |x_n|^p\right)^{1/p}$$

is a norm for $\{x_n\} \in \ell^p$, $1 < p < \infty$, and show that

$$\|\{x_n\}\|_\infty = \sup_n |x_n|$$

is a norm for $\{x_n\} \in \ell^\infty$.

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A function $f : X \to \mathbb{R}$ is a *norm* if (i) $f(x) \geq 0$, (ii) $f(x) = 0$ if and only if $x = 0$, (iii) $f(\alpha x) = |\alpha| f(x)$, and (iv) $f(x+y) \leq f(x) + f(y)$.

(i) is trivial in both cases.

(ii) is easy: if $(\sum_n |x_n|^p)^{1/p} = 0$ then $\sum_n |x_n|^p = 0$ then $|x_n| = 0$ for every n then $\{x_n\} = \{0\}$; and the other direction is trivial. Similarly, we know that $|x_n| \geq 0$ regardless of $\{x_n\}$, and so if $\sup_n |x_n| = 0$, we have $0 \leq |x_0| \leq 0$, that is $x_n = 0$, for every n.

(iii) is trivial: $\left(\sum_n |\alpha x_n|^p\right)^{1/p} = \left(|\alpha|^p \sum_n |x_n|^p\right)^{1/p} = |\alpha| \left(\sum_n |x_n|^p\right)^{1/p}$.

(iv), the triangle inequality, is the most difficult of these four, and is also easy. One could do an immense amount of work, but one doesn't need to: the triangle inequality for ℓ^p, $1 < p < \infty$, is the series version of the Minkowski inequality, Ex. 9.3.

The triangle inequality for ℓ^∞ is easier:

$$\begin{aligned}\|\{x_n + y_n\}\|_\infty &= \sup_n |x_n + y_n| \\ &\le \sup_n (|x_n| + |y_n|) \\ &\le \sup_n |x_n| + \sup_n |y_n| \\ &= \|\{x_n\}\|_\infty + \|\{y_n\}\|_\infty.\end{aligned}$$

(4) Show that in the space $C_{\mathbb{R}}([0,1])$ the norms

$$\|f\|_1 = \int_0^1 (1-t)|f(t)|\,dt$$

and

$$\|f\|_2 = \int_0^1 (1-t^3)|f(t)|\,dt$$

are equivalent. (See Definition 4.4. In this and the next problem, you don't need to prove these functions are norms.)

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Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there is a constant C so that

$$\frac{1}{C}\|f\|_1 \le \|f\|_2 \le C\|f\|_1$$

for every f; in our case for every $f \in C_{\mathbb{R}}([0,1])$, the space of all functions continuous and defined on $[0,1]$. Now this claim is

$$\begin{aligned}\frac{1}{C}\int_0^1 (1-t)|f(t)|\,dt &\le \int_0^1 (1-t^3)|f(t)|\,dt \\ &\le C\int_0^1 (1-t)|f(t)|\,dt,\end{aligned}$$

and this holds if we can find a constant C so that

$$\frac{1}{C}(1-t) \le (1-t^3) \le C(1-t)$$

for all $t \in [0,1]$. (Actually, we just need to find a C so this holds for all $t \in (0,1)$; two points are a set of zero measure and don't affect the value of the integral.) The first of these inequalities is $\frac{1}{C}(1-t) \le 1 - t^3$, and simplifies to

$$1/C \le \frac{1-t^3}{1-t} = \frac{(1-t)(1+t+t^2)}{1-t} = 1 + t + t^2.$$

This holds for all $t \in (0,1)$ if $C \geq 1$. The second inequality is $(1-t^3) \leq C(1-t)$ or

$$C \geq \frac{1-t^3}{1-t} = 1 + t + t^2,$$

which holds for any $t \in (0,1)$ if $C \geq 3$. We choose $C = 3$, and have

$$\frac{1}{3}\|f\|_1 \leq \|f\|_2 \leq 3\|f\|_1$$

for all $f \in C_{\mathbb{R}}([0,1])$.

(Note that, if we were in the mood, we could write that as $\frac{1}{3}\|f\|_1 \leq \|f\|_2$ and $\|f\|_2 \leq 3\|f\|_1$, divide those to get $\|f\|_1 \leq 3\|f\|_2$ and $\frac{1}{3}\|f\|_2 \leq \|f\|_1$, and combine these as

$$\frac{1}{3}\|f\|_2 \leq \|f\|_1 \leq 3\|f\|_2.$$

When we prove equivalency, it doesn't matter which norm is "in the middle".)

(5) Let $P([0,1])$ be the space of polynomials defined on $[0,1]$. Show that the norms

$$\|p\|_a = \sup\{|p(x)| \mid x \in [0,1]\}$$

and

$$\|p\|_b = \int_0^1 |p(x)|\, dx,$$

where $p \in P([0,1])$, are not equivalent.

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To show the norms are not equivalent, we have to show there is no C so that

$$\frac{1}{C}\|p\|_a \leq \|p\|_b \leq C\|p\|_a$$

for all $p \in P([0,1])$. There are two ways to do this; two that immediately come to mind, that is. The first would be to find a $p \in P([0,1])$ so that one of its norms is finite and the other infinite. The second would be to find a sequence p_n so that one of the norms is a constant (or bounded) and the other increases or decreases without bounds. We use this second way: Let $p_n : [0,1] \to \mathbb{R}$ be defined as

$$p_n(x) = (1-x)^n.$$

Very clearly this is a polynomial; we do not want to expand it. This is a function that has its maximum at $x = 0$ with $p_n(0) = 1$, and for which $\int_0^1 p_n(x)\, dx = 1/(n+1)$. This is to say

$$\|p_n\|_b = 1/(n+1),$$

but

$$\|p_n\|_a = 1,$$

so any C that would show equivalency would have to have

$$\frac{1}{C} \leq 1/(n+1) \leq C$$

for every $n = 1, 2, \ldots$, which is impossible: for any fixed C, we can always find a bigger n so that the first inequality does not hold.