Analysis IV
Spring 2011
Exercises 11

(1) Prove Theorem 4.7 (b) and (c).

(2) Let
$$c = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \lim_{n\to\infty} x_n \text{ exists}\}$$
and
$$c_0 = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \lim_{n\to\infty} x_n = 0\}.$$
Prove true or false.

(i) $c_0 \subset \ell^1$

(ii) If $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^{p/(p-1)}$, $1 < p < \infty$, then $\{x_n y_n\} \in \ell^1$.

(iii) If $x \in c$, there exists such $y \in c$ that $x + y \in c_0$.

(3) Let c and c_0 be as above, and let
$$c_{0,0} = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \exists N \in \mathbb{N} \text{ so that } x_n = 0 \ \forall n > N\}.$$
Prove that $c_{0,0} \subset \ell^p \subset \ell^\infty$ $(1 \le p < \infty)$ and $c_{0,0} \subset c_0 \subset c \subset \ell^\infty$.

(4) Let X be a space with a norm $\|\cdot\|_X$. Let $x \in X \setminus \{0\}$ and $r \in \mathbb{R}$, $r > 0$. Find a scalar $\alpha \in \mathbb{R}$ so that
$$\|\alpha x\|_X = r.$$

(5) Draw the unit circles defined by the following norms:
$$\|x\|_1 = |x_1| + |x_2|,$$
$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2} \text{ and}$$
$$\|x\|_3 = |x_1| + 3|x_2|,$$
where $x \in \mathbb{R}^2$. (The unit circle defined by $\|\cdot\|$ consists of those points x for which $\|x\| = 1$.)