Analysis IV
Spring 2011
Exercises 11 / Answers

(1) Prove Theorem 4.7 (b) and (c):

> **Theorem 4.7** Let X be a vector space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on X. Let d_1 and d_2 be metrics defined by $d_1(x,y) = \|x-y\|_1$ and $d_2(x,y) = \|x-y\|_2$. Let $\{x_n\} \in X$ be a sequence.
> (a) $\{x_n\}$ converges to x in the metric space (X, d_1) if and only if $\{x_n\}$ converges to x in the metric space (X, d_2).
> (b) $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_1) if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_2).
> (c) (X, d_1) is complete if and only if (X, d_2) is complete.

* * *

(b) Let $\{x_n\}$ be a Cauchy sequence in (X, d_1), that is, let there be for every $\epsilon_1 > 0$ a number N so that $d_1(x_m, x_n) < \epsilon_1$ if $m, n > N$.

We will show the Cauchy condition holds for $\{x_n\}$, d_2 and $\epsilon_2 > 0$. Since the norms are equivalent, there is some constant C so that

$$\begin{aligned} d_2(x_m, x_n) &= \|x_m - x_n\|_2 \\ &\le C\|x_m - x_n\|_1 = Cd_1(x_m, x_n), \end{aligned}$$

and since $\{x_n\}$ is a Cauchy sequence in d_1, for some N we have (with $\epsilon_1 = \epsilon_2/C$) the inequality

$$Cd_1(x_m, x_n) \le C\frac{\epsilon_2}{C} = \epsilon_2$$

when $m, n > N$.

(c) Assume that (X, d_1) is complete, that is, assume that every Cauchy sequence in d_1 converges in d_1. Let $\{x_n\}$ be a Cauchy sequence in (X, d_2). By (b), $\{x_n\}$ is a Cauchy sequence in (X, d_1). Because (X, d_1) is complete, $\{x_n\}$ converges in (X, d_1). By (a), $\{x_n\}$ converges in (X, d_2).

(2) Let

$$c = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \lim_{n\to\infty} x_n \text{ exists}\}$$

and

$$c_0 = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \lim_{n\to\infty} x_n = 0\}.$$

Prove true or false.

(i) $c_0 \subset \ell^1$
(ii) If $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^{p/(p-1)}$, $1 < p < \infty$, then $\{x_n y_n\} \in \ell^1$.
(iii) If $x \in c$, there exists such $y \in c$ that $x + y \in c_0$.

* * *

(i) $c_0 \subset \ell^1$: False. Clearly $\{1/n\}_{n=1}^{\infty} \in c_0$, but

$$\sum_{n=1}^{\infty} 1/n = \infty,$$

so $\{1/n\} \notin \ell^1$.

(ii) If $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^{p/(p-1)}$, $1 < p < \infty$, then $\{x_n y_n\} \in \ell^1$: True. For $\{x_n y_n\}$, we have by Hölder's inequality that

$$\sum_{n=1}^{\infty} x_n y_n \leq \left(\sum_{n=1}^{\infty} x_n^p \right)^{1/p} \left(\sum_{n=1}^{\infty} x_n^{p/(p-1)} \right)^{(p-1)/p} < \infty.$$

(iii) If $x \in c$, there exists such $y \in c$ that $x + y \in c_0$: True. Let $\{x_n\} \in c$. Then $\{-x_n\} \in c$, and $\{x_n\} + \{-x_n\} = \{0\} \in c_0$.

(3) Let c and c_0 be as above, and let

$$c_{0,0} = \{\{x_n\}_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \exists N \in \mathbb{N} \text{ so that } x_n = 0 \ \forall n > N\}.$$

Prove that $c_{0,0} \subset \ell^p \subset \ell^\infty$ $(1 \leq p < \infty)$ and $c_{0,0} \subset c_0 \subset c \subset \ell^\infty$.

* * *

(a) $c_{0,0} \subset \ell^p$ $(1 \leq p < \infty)$

Let $\{x_n\} \in c_{0,0}$. Then for some N

$$\left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} = \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p},$$

which is finite because each x_n is finite.

(b) $\ell^p \subset \ell^\infty$ $(1 \leq p < \infty)$

If $\{x_n\} \in \ell^p$, then $\sum_{n=1}^{\infty} |x_n|^p < \infty$. This means that $|x_n| < \infty$ for each n, and $\lim_{n \to \infty} |x_n| = 0$. Thus there is a N so that $|x_n| < 1$ when $n > N$, and so

$$|x_n| \leq \max\{|x_1|, \ldots, |x_N|, 1\} < \infty.$$

Because this is one upper limit for $|x_n|$ and the supremum is the smallest upper limit,

$$\sup_n |x_n| < \infty.$$

(Just noting $|x_n| < \infty$ for each n is not enough. That holds for $\{n\}_{n=1}^{\infty} \notin \ell^\infty$ too.)

(c) $c_{0,0} \subset c_0$

Obvious, since if $\{x_n\} \in c_{0,0}$, then for some N

$$\lim_{n\to\infty} x_n = \lim_{\substack{n\to\infty \\ n>N}} 0 = 0.$$

(d) $c_0 \subset c$

Also obvious, because if the limit exists and is zero, the limit exists. (In addition to obvious, slightly dim as well.)

(e) $c \subset \ell^\infty$

Let $\{x_n\} \in c$. Then $x = \lim_{n\to\infty} x_n$ exists and (though this could have been made clearer in the definition) $x \in \mathbb{R}$, that is, $|x| < \infty$. Because $x_n \in \mathbb{R}$ for every n, we know $|x_n| < \infty$ for all n as well. Since $\{x_n\}$ converges to x, we can have $|x_n - x| < 1$ with every $n > N$ for some N, and by the usual argument,

$$|x_n| \le \max\{|x_1|, \ldots, |x_N|, |x| + 1\} < \infty.$$

(4) Let X be a space with a norm $\|\cdot\|_X$. Let $x \in X \setminus \{0\}$ and $r \in \mathbb{R}$, $r > 0$. Find a scalar $\alpha \in \mathbb{R}$ so that

$$\|\alpha x\|_X = r.$$

* * *

Because $x \neq 0$, we know that $\|x\| \neq 0$. Choose $\alpha = r/\|x\| \in \mathbb{R}$. Then

$$\|\alpha x\| = |\alpha|\|x\| = \frac{r\|x\|}{\|x\|} = r.$$

(5) Draw the unit circles defined by the following norms:

$$\begin{aligned} \|x\|_1 &= |x_1| + |x_2|, \\ \|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2} \text{ and} \\ \|x\|_3 &= |x_1| + 3|x_2|, \end{aligned}$$

where $x \in \mathbb{R}^2$. (The unit circle defined by $\|\cdot\|$ consists of those points x for which $\|x\| = 1$.)

* * *

The unit circles can be found by solving $\|x\| = 1$ for x. That is, for example, in the case of the third norm, solving $|x_1| + 3|x_2| = 1$ for (x_1, x_2). The results are as follows:

- For $\|\cdot\|_1$, a diamond shape (*vinoneliö*) with corners at $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.
- For $\|\cdot\|_2$, the "normal" circle of radius 1 and center $(0, 0)$.
- For $\|\cdot\|_3$, an elongated diamond (*virutettu vinoneliö*?) with corners at $(1, 0)$, $(0, 1/3)$, $(-1, 0)$ and $(0, -1/3)$.