Analysis IV
Spring 2011
Exercises 12 / Answers

(1) Prove the second example on p. 30 of the lectures: If $f, g \in L^2$, then $fg \in L^1$ and

$$< f, g >= \int fg \, dm$$

is an inner product.

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Definition. (abbreviated.) A function $< \cdot, \cdot >: X \times X \to \mathbb{R}$ is an *inner product*, if the following four conditions hold for $x, y \in X$:

(a) $< x, x > \geq 0$ (note: for $< x, x >$, not $< x, y >$!),

(b) $< x, x >= 0$ if and only if $x = 0$,

(c) $< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$ and

(d) $< x, y >=< y, x >$.

In this particular case, we deal with a possible inner product defined for functions $f, g, h \in L^2$, that is, are functions so that $\int |f|^2 \, dm < \infty$ and the same for g and h. Remember that (as by Theorem 3.4) for L^2 we write $f = g$ if $f = g$ a.e.; we treat two functions as the same if they are different only in a set of zero measure.[1] Here the conditions take the following form

(a) $\int ff \, dm = \int |f|^2 \, dm \geq 0$ (clear because $|f|^2 \geq 0$)

(b) If $\int |f|^2 \, dm = 0$, then $|f|^2 = 0$ a.e. and $f = 0$ a.e.; and in L^2, that means $f = 0$. If, on the other hand, $f = 0$, then clearly $\int |f|^2 \, dm = 0$.

(c) By the basic properties of the integral (see chapter 2 for the theorems),

$$\begin{aligned} < \alpha f + \beta g, h > &= \int (\alpha f + \beta g) g \, dm \\ &= \alpha \int fh \, dm + \beta \int gh \, dm \\ &= \alpha < f, h > + \beta < g, h > . \end{aligned}$$

[1] Why yes, this does mean that $f = g$ if $f = 0$ and

$$g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

This "makes sense" because the two functions have no difference that the integral L^2 norm can "see".

(d) Clearly

$$\int fg\,dm = \int gf\,dm.$$

Finally, we have to show that "if $f, g \in L^2$, then $fg \in L^1$". This is pretty important, since if we don't know that $fg \in L^1$ we could be dealing with a function $< \cdot, \cdot >$ that is not well defined; an infinity would not be difficult, but the integral being undefined would be really, really bad. But we know that "if $f, g \in L^2$, then $fg \in L^1$", because this is just Hölder's inequality with $p = q = 2$.

(2) Prove the third example on p. 30 of the lectures: If $a = \{a_n\}, b = \{b_n\} \in \ell^2$, then $\{a_n b_n\} \in \ell^1$ and

$$< a, b > = \sum_{n=1}^{\infty} a_n b_n$$

is an inner product.

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First, "if $a = \{a_n\}, b = \{b_n\} \in \ell^2$, then $\{a_n b_n\} \in \ell^1$". This holds by the same as in the previous problem: by Hölder's inequality, though in this case for series.

Second, the four conditions of an inner product go as follows, for $a, b, c \in \ell^2$:

(a) $\sum_{n=1}^{\infty} a_n^2 \geq 0$ because $a_n^2 \geq 0$ for each n,

(b) If $\sum_{n=1}^{\infty} a_n^2 = 0$ then $a_n^2 = 0$ for each n because each a_n^2 is non-negative; and so $a_n = 0$ for each n, that is,[2] $\{a_n\} = 0$. If $\{a_n\} = 0$, well, obviously $\sum_{n=1}^{\infty} a_n^2 = 0$.

(c) Now

$$< \alpha a + \beta b, c > = \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) c_n$$

$$\leq \alpha \sum_{n=1}^{\infty} a_n c_n + \beta \sum_{n=1}^{\infty} b_n c_n = \alpha < a, c > + \beta < b, c > .$$

[2] It is convenient, though potentially confusing, that we can write $\{a_n\} = 0$. Remember that this means "the sequence of the numbers a_n, where n is the index, is the same as the zero-of-ℓ^2, that is, the sequence of zero, zero, zero, and so on". We don't write $\{a_n\}_{n=1}^{\infty}$ because we hope the index is clear from the context, and likewise don't write $\{0\}_{n=1}^{\infty}$ because we hope a reasonable reader would presume that a sequence can only be equal to a sequence, so 0 must stand for the zero of the space $\{a_n\}$ is in; which is, the zero of ℓ^2, which is the sequence of zeros. See Exercises 9 for more ℓ^p funtime.

The same principle is at work when a mathematician write $\sum$ or $\sum_n$ for $\sum_{n=1}^{\infty}$: she is leaving things out because of laziness, and because she hopes the audience will understand what is meant, and concentrate on the important bits instead of notation details.

Note that to separate the sums we need to know that $\sum_n a_n c_n$ and $\sum_n b_n c_n$ are well defined; we know this because $ac, bc \in \ell^1$.

(d) Clearly

$$< a, b >= \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} b_n a_n =< b, a > .$$

(3) Prove Lemma 5.6: Let X be an inner product space with inner product $< \cdot, \cdot >$. Then for all $u, v, x, y \in X$,

(a) $< u+v, x+y > - < u-v, x-y >= 2 < u, y > +2 < v, x >$,
(b) for complex X,

$$\begin{aligned} 4 < u, y > &=< u + v, x + y > - < u - v, x - y > \\ &+ i < u + iv, x + iy > -i < u - iv, x - iy > . \end{aligned}$$

* * *

(a) Brute calculation; here (c) and (d) stand for the third and fourth properties of an inner product:

$$\begin{aligned} &< u + v, x + y > - < u - v, x - y > \\ &\overset{(c)}{=}< u, x + y > + < v, x + y > - < u, x - y > + < v, x - y > \\ &\overset{(d)}{=}< x + y, u > + < x + y, v > - < x - y, u > + < x - y, v > \\ &\overset{(c)}{=}< x, u > + < y, u > + < x, v > + < y, v > \\ &\quad - < x, u > + < y, u > + < x, v > - < y, v > \\ &=< y, u > +2 < x, v > + < y, u > \\ &\overset{(d)}{=} 2 < u, y > +2 < v, x > \end{aligned}$$

(b) Let us take the four inner products on the right-hand side and calculate them separately:

For the first part, we use (a) above:

$$\begin{aligned} &< u + v, x + y > \\ &= 2 < u, y > +2 < v, x > + < u - v, x - y > . \end{aligned}$$

We keep the second part as it is:

$$- < u - v, x - y > .$$

For the third, we use (a):

$$\begin{aligned} &i < u + iv, x + iy > \\ &= 2i < u, iy > +2i < iv, x > +i < u - iv, x - iy > . \end{aligned}$$

And we keep the fourth part as it is:

$$-i < u - iv, x - iy > .$$

Adding these four together, this much remains:

$$2 < u,y > +2 < v,x > +2i < u,iy > +2i < iv,x >$$
$$= 2 < u,y > +2 < v,x > +2i\overline{< iy,u >} + 2\underbrace{i^2}_{=-1} < v,x >$$
$$= 2 < u,y > +2 \underbrace{i\bar{i}}_{=-i^2=1} < u,y >$$
$$= 4 < u,y > .$$

(4) Prove Theorem 5.7: Let X be an inner product space with inner product $< \cdot,\cdot >$ and induced norm $\|\cdot\|$. Then for all $x,y \in X$,

(a) the parallelogram rule (*suunnikassääntö*):
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$
(b) for real X, $4 < x,y >= \|x+y\|^2 - \|x-y\|^2$,
(c) for complex X, the polarization identity:
$$4 < x,y > = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

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(a) Remember that $\|x\| = \sqrt{< x,x >}$. Then
$$\|x+y\|^2 + \|x-y\|^2$$
$$=< x+y,x+y > + < x-y,x-y >,$$
and by Lemma 5.4 (c),
$$= < x,x > + < x,y > +\overline{< x,y >} + < y,y >$$
$$+ < x,x > - < x,y > -\overline{< x,y >} + < y,y >$$
$$=2 < x,x > +2 < y,y >$$
$$=2(\|x\|^2 + \|y\|^2).$$

(b) Here as in (a), except with $< x,y >= \overline{< x,y >}$,
$$\|x+y\|^2 =< x,y > +2 < x,y > + < y,y >$$
and
$$\|x-y\|^2 < x,x > -2 < x,y > + < y,y >,$$
and substracting the second from the first gives
$$\|x+y\|^2 - \|x-y\|^2 = 4 < x,y > .$$

(c) Here the difference is that the condition (d) for an inner product is $< x,y >= \overline{< y,x >}$ and not $< x,y >=< y,x >$. Thus
$$\|x+y\|^2 =< x,x > + < x,y > + < y,x > + < y,y >$$
and
$$\|x-y\|^2 =< x,x > - < x,y > - < y,x > + < y,y >,$$

so

$$\|x+y\|^2 - \|x-y\|^2 = 2 < x,y > +2 < y,x > . \qquad (1)$$

(Note that this is not the equality of (b), because for complex inner products $< x,y > \neq < y,x >$.) Next, by Lemma 5.4 (c) we have

$$\begin{aligned}\|x+iy\|^2 &= < x+iy, x+iy > \\ &= < x,x > -i < x,y > +i < y,x > - < y,y >\end{aligned}$$

and

$$\begin{aligned}\|x-iy\|^2 &= < x-iy, x-iy > \\ &= < x,x > +i < x,y > -i < y,x > - < y,y > .\end{aligned}$$

Thus

$$\begin{aligned}i\|x+iy\|^2 - i\|x-iy\|^2 &= i(-2i < x,y > +2i < y,x >) \\ &= 2 < x,y > -2 < y,x >,\end{aligned}$$

and adding this to (1) gives

$$\begin{aligned}&\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \\ &= 4 < x,y > .\end{aligned}$$

(5) Let $1 \leq p < q < \infty$. Prove that $\ell^p \subset \ell^q$. (Hint: start with those $x \in \ell^p$ for which $\|x\|_p = 1$.)

$$* * *$$

Let us assume $\|x\|_p = 1$, that is,

$$\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} = 1.$$

Then $\sum_{n=1}^{\infty} |x_n|^p = 1$, and $|x_n|^p \leq 1$ for each n. (To assume otherwise would be an instant contradiction.)

Because $|x_n|^p \leq 1$ for all n for some $1 < p < \infty$, then $|x_n| \leq 1$ for all n. By this,

$$|x_n|^q < |x_n|^p$$

for $1 < p < q < \infty$ and all n. Because $x \in \ell^p$, we know that

$$\sum_{n=1}^{\infty} |x_n|^q \leq \sum_{n=1}^{\infty} |x_n|^p < \infty,$$

so

$$\left(\sum_{n=1}^{\infty} |x_n|^q\right)^{1/q} < \infty,$$

that is, $x \in \ell^q$.

Next, if $x \in \ell^p$ and $\|x\|_p = 0$, then $x = 0$, and $x \in \ell^q$.

Now, if $x \in \ell^p$ and $\|x\|_p \neq 0, 1$, we can take the sequence $y = x/\|x\|_p \in \ell^p$, for which

$$\|y\|_p = \left\| \frac{x}{\|x\|_p} \right\|_p = \frac{1}{\|x\|_p} \|x\|_p = 1.$$

By the preceding, $y \in \ell^q$. Since $x \in \ell^p$, we know $0 < \|x\|_p < \infty$. Thus

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} |x_n|^q \right)^{1/q} &= \left(\sum_{n=1}^{\infty} \left| x_n \frac{\|x\|_p}{\|x\|_p} \right|^q \right)^{1/q} \\
&= \left(\|x\|_p^q \sum_{n=1}^{\infty} \left| x_n \frac{1}{\|x\|_p} \right|^q \right)^{1/q} \\
&= \|x\|_p \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q} < \infty,
\end{aligned}$$

and we have $x \in \ell^q$ as well.