Analysis IV
Spring 2011
Exercises 13 / Answers

(1) Let the angle between $x, y \in \ell^2$ be defined as

$$\theta(x,y) = \cos^{-1}\left(\frac{< x, y >}{\|x\|\|y\|}\right)$$

with the standard inner product and norm of ℓ^2.

Let $x = \{1/2^n\}_{n=1}^{\infty}$ and $y = \{C/3^n\}_{n=1}^{\infty}$, where $C \in \mathbb{R}$. For what value of C does $\theta(x,y) = \pi/3$? For which value of C does $\theta(x,y) = \pi/2$?

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For the $\pi/3$ case, we need to solve

$$\pi/3 = \cos^{-1}\left(\frac{< x, y >}{\|x\|\|y\|}\right)$$

or

$$\frac{< x, y >}{\|x\|\|y\|} = \frac{1}{2}. \tag{1}$$

Here

$$< x, y >= \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{C}{2^n 3^n} = C \sum_{n=1}^{\infty} \frac{1}{6^n} = \frac{C}{5}.$$

Because

$$\|x\|^2 = \sum_{n=1}^{\infty} x_n^2 = \sum_{n=1}^{\infty} \frac{1}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{4^n} = 1/3$$

and

$$\|y\|^2 = \sum_{n=1}^{\infty} \frac{C^2}{(3^n)^2} = C^2 \sum_{n=1}^{\infty} \frac{1}{9^n} = C^2 1/8,$$

we have $\|x\| = 1/\sqrt{3}$ and $\|y\| = C/\sqrt{8}$. Thus, the equation (1) becomes

$$\frac{1}{2} = \frac{C/5}{\sqrt{1/3}\,C\sqrt{1/8}} = \frac{\sqrt{24}}{5} \quad (\approx 0.98).$$

This is not soluble[1] since C is eliminated and what remains is not equal; thus the answer to both questions is ”for no value of C”.

[1]”soluble” means ”having a solution”, ”being something that can be solved”. For example, $\frac{1}{9}x^5 - \frac{9}{5}x^3 + \frac{5}{\pi}x^2 = 0$ no doubt is soluble for $x \in \mathbb{R}$, though I have no idea what the solution might be.

For $\pi/2$ the necessary equation would be $0 =< x, y > /(\|x\|\|y\|)$, so the answer remains the same. (If $< x, y >= 0$ then $C = 0$ then $\|y\| = 0$, and then a divide by zero error.)

(2) What is the angle between x^2 and x? Use the standard inner product and norm of $L^2([0, 1])$.

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By the same definition of an angle as above,

$$\|x\|^2 = \int_0^1 x^2\, dx = \Big/_0^1 \frac{1}{3}x^3 = \frac{1}{3}$$

so $\|x\| = 1/\sqrt{3}$, and

$$\|x^2\|^2 = \int_0^1 (x^2)^2\, dx = \Big/_0^1 \frac{1}{5}x^5 = \frac{1}{5}$$

so $\|x^2\| = 1/\sqrt{5}$, and

$$< x, y >= \int_0^1 xx^2\, dx = \Big/_0^1 \frac{1}{4}x^4 = \frac{1}{4}.$$

Thus

$$\theta(x, x^2) = \cos^{-1} \frac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \cos^{-1} \frac{\sqrt{15}}{4} \approx 0.08\pi.$$

(3) Let $T : C_{\mathbb{R}}([0, 1]) \to \mathbb{R}$ be defined by

$$T(f) = \int_0^1 f(x)\, dx.$$

Show that T is continuous.

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By Lemma 6.1 (d), T is continuous if there is a $k > 0$ so that $\|T(f)\| \leq k$ for every f with $\|f\| \leq 1$. Here $\|f\| \leq 1$ is

$$\|f\|_{C_{\mathbb{R}}([0,1])} = \sup_{x\in[0,1]} f(x) \leq 1,$$

and if this condition holds, then

$$\|T(f)\|_{\text{real numbers}} = \left|\int_0^1 f(x)\, dx\right| \leq |\int_0^1 1\, dx| = 1.$$

We choose $k = 1$, and Lemma 6.1 (d) holds.

(4) Let $h \in L^\infty([0, 1])$. Show that $T : L^2([0, 1]) \to L^2([0, 1])$,

$$T(f) = hf,$$

is continuous.

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Since $h \in L^\infty([0,1])$, we know that $\operatorname{ess\,sup}_{[0,1]} h < \infty$, that is, we know there is some $M > 0$ so that $h(x) \le M$ for all $x \in [0,1]\backslash A$, where $m(A) = 0$. Thus

$$\begin{aligned}\|T(f)\|^2 &= \int_{[0,1]} (hf)^2\, dm \\ &= \int_{[0,1]\backslash A} (hf)^2\, dm \\ &\le \int_{[0,1]\backslash A} (Mf)^2\, dm \\ &= M^2 \int_{[0,1]} f^2\, dm = M^2\|f\|^2,\end{aligned}$$

and T is continuous by Lemma 6.1 (e), $k = M$.

(5) Show that if $(x_1, x_2, \ldots) \in \ell^2$, then

$$(0, 4x_1, x_2, 4x_3, x_4, 4x_5, x_6, \ldots) \in \ell^2.$$

Let $T : \ell^2 \to \ell^2$ be defined by

$$T(x_1, x_2, \ldots) = (0, 4x_1, x_2, 4x_3, x_4, 4x_5, x_6, \ldots).$$

Show that T is continuous.

* * *

If $(x_1, x_2, \ldots) \in \ell^2$, then

$$\sum_{n=1}^{\infty} x_n^2 < \infty.$$

Now $(0, 4x_1, x_2, 4x_3, x_4, \ldots) \in \ell^2$, because

$$\begin{aligned}0 &\le 0 + (4x_1)^2 + x_2^2 + (4x_3)^2 + x_4^2 + (4x_5)^2 + x_6^2 + \ldots \\ &\le 0 + (4x_1)^2 + (4x_2)^2 + (4x_3)^2 + (4x_4)^2 + (4x_5)^2 + (4x_6)^2 + \ldots,\end{aligned}$$

and

$$\sum_{n=1}^{\infty} (4x_n)^2 = 16 \sum_{n=1}^{\infty} x_n^2 < \infty.$$

Next, to show that T is continuous — well, the lines above show that $\|T(x)\|^2 \le 16\|x\|^2$, or

$$\|T(x)\| \le 4\|x\|,$$

and by Lemma 6.1 (e) we are done.

The second exam is from the beginning of Chapter 3 to Theorem 6.5, and the exercises from the sixth problem of Exercises 7 to the last of Exercises 13.