Analysis IV
Spring 2011
Exercises 14 ($\Leftrightarrow$ Second exam)

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(1) Let f be defined as $f : [0,1] \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. Calculate the norm of f in

(a) $C_{\mathbb{R}}([0,1])$ and
(b) $L^p([0,1])$, where $1 \leq p < \infty$.

(2) Let $z_n = (1+i)n^{-1/3}$. Show that $\{z_n\} \in \ell^p(\mathbb{C})$, when $p > 3$ and $\{z_n\} \notin l^3(\mathbb{C})$.

(3) Let

$$\mathcal{P} = \{x \mid x \text{ is a polynomial with real coefficients } \}.$$

Define the inner product in $\mathcal{P}$ by setting

$$< x, y >= \int_0^1 x(t)y(t)\,dt.$$

Let $x_1(t) = 1$, $x_2(t) = a + t$ and $x_3(t) = b + ct + t^2$.

(i) Calculate the inner products $< x_1, x_2 >$, $< x_1, x_3 >$ and $< x_2, x_3 >$.
(ii) Determine a, b and c such that $\{x_1, x_2, x_3\}$ is an orthogonal set, i.e., polynomials x_1, x_2, x_3 are orthogonal to each other.

(4) Let $\{x_n\}$ be a sequence of vectors in a Hilbert space such that

$$\sum_{k=1}^{\infty} \|x_k\| < \infty,$$

and define

$$y_n = \sum_{k=1}^{n} x_k.$$

Prove that $\{y_n\}$ is a Cauchy sequence.

(5) Let $T : C_{\mathbb{R}}([0,1]) \to \mathbb{R}$ be defined by

$$T(f) = \int_0^1 f(x)\,dx.$$

Show that T is continuous.