Analysis IV
Spring 2011
Exercises 14 ($\Leftrightarrow$ Second exam) / Answers

(1) Let f be defined as $f : [0,1] \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$. Calculate the norm of f in

(a) $C_{\mathbb{R}}([0,1])$ and
(b) $L^p([0,1])$, where $1 \leq p < \infty$.

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See Exercises 10 problem 2.

(2) Let $z_n = (1+i)n^{-1/3}$. Show that $\{z_n\} \in \ell^p(\mathbb{C})$, when $p > 3$ and $\{z_n\} \notin l^3(\mathbb{C})$.

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Note that we deal with the complex ℓ^p space $\ell^p(\mathbb{C})$. Thus i is the complex unit, or the number for which $i^2 = -1$, and $|\cdot|$ stands for the complex absolute value,

$$|x + iy| = (x^2 + y^2)^{1/2}.$$

To see whether $\{z_n\} \in \ell^p$, we need to see if

$$\sum_{n=1}^{\infty} |z_n|^p$$

is finite (then $\{z_n\}$ is in ℓ^p) or infinite (then $\{z_n\}$ is not in ℓ^p). Now

$$\begin{aligned} \sum_{n=1}^{\infty} |z_n|^p &= \sum_{n=1}^{\infty} |1+i|^p n^{-p/3} \\ &= (1^2 + 1^2)^{p/2} \sum_{n=1}^{\infty} n^{-p/3} \\ &= 2^{p/2} \sum_{n=1}^{\infty} n^{-p/3}, \end{aligned}$$

and we know from earlier analysis courses that $\sum_n (1/n)^{p/3}$ converges if $p/3 > 1$, and diverges when $p/3 \leq 1$. This is to say, $\sum_n (1/n)^{p/3} < \infty$ when $p > 3$, and $\sum_n n^{-p/3} = \infty$ if $p = 3$.

Finally, to show that $\{z_n\} \in \ell^\infty(\mathbb{C})$, we have to show that $\sup_n |z_n| < \infty$. This is clear, because

$$\sup_n |z_n| = \sqrt{2} \underbrace{\sup_n n^{-1/3}}_{=1} = \sqrt{2} < \infty.$$

(3) Let

$$\mathcal{P} = \{x \mid x \text{ is a polynomial with real coefficients }\}.$$

Define the inner product in $\mathcal{P}$ by setting

$$< x, y >= \int_0^1 x(t)y(t)\, dt.$$

Let $x_1(t) = 1$, $x_2(t) = a + t$ and $x_3(t) = b + ct + t^2$.

(i) Calculate the inner products $< x_1, x_2 >$, $< x_1, x_3 >$ and $< x_2, x_3 >$.

(ii) Determine a, b and c such that $\{x_1, x_2, x_3\}$ is an orthogonal set, i.e., polynomials x_1, x_2, x_3 are orthogonal to each other.

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(i) The inner products are:

$$\begin{aligned}
< x_1, x_2 > &= \int_0^1 a + t\, dt = a + 1/2, \\
< x_1, x_3 > &= \int_0^1 b + ct + t^2\, dt = b + c/2 + 1/3 \text{ and} \\
< x_2, x_3 > &= \int_0^1 (a + t)(b + ct + t^2)\, dt \\
&= \int_0^1 ab + (ac + b)t + (a + c)t^2 + t^3\, dt \\
&= ab + (ac + b)/2 + (a + c)/3 + 1/4.
\end{aligned}$$

(ii) Two objects x and y are defined to be orthogonal if $< x, y >= 0$. We choose a, b and c so that the three inner products are all equal to zero. This is,

$$\begin{cases} a + 1/2 & = 0 \\ b + c/2 + 1/3 & = 0 \\ ab + (ac + b)/2 + (a + c)/3 + 1/4 & = 0. \end{cases}$$

This gives $a = -1/2$, $b = 1/6$ and $c = -1$.

(4) Let $\{x_n\}$ be a sequence of vectors in a Hilbert space such that

$$\sum_{k=1}^{\infty} \|x_k\| < \infty, \tag{1}$$

and define

$$y_n = \sum_{k=1}^{n} x_k.$$

Prove that $\{y_n\}$ is a Cauchy sequence.

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A "Hilbert space" is a complete inner product space: it has objects for which we have an inner product $< \cdot, \cdot >$, it has a norm $\| \cdot \|$ defined as

$$\|x\| = \sqrt{< x, x >}$$

(see Lemma 5.5), and a metric $d(\cdot, \cdot)$ defined by that norm,

$$d(x, y) = \|x - y\|$$

(see Lemma 4.2). Finally, because the space is complete, all Cauchy sequences of such objects (in that metric) converge. This all so far is good to know, but not terribly important for this problem.

To prove that $\{y_n\}$ is a Cauchy sequence, let us take two elements from it, y_n and y_m, and show that for any $\epsilon > 0$ we have

$$d(y_n, y_m) = \|y_n - y_m\| < \epsilon$$

if $m, n > N$ for some sufficiently large N.

We can assume $n > m$. Now

$$\|y_n - y_m\| = \left\| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right\| = \left\| \sum_{k=m+1}^{n} x_k \right\| .$$

Next, we use the triangle inequality $n - m - 1$ times, and get

$$\left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\|.$$

Because of (1), we know that if m is sufficiently large, we can get $\sum_{k=m+1}^{\infty} \|x_k\|$ to be as small as we want — which means, for any $\epsilon > 0$ we can find a N so that

$$\|y_n - y_m\| < \epsilon$$

when $n, m > N$.

(5) Let $T : C_{\mathbb{R}}([0,1]) \to \mathbb{R}$ be defined by

$$T(f) = \int_0^1 f(x)\,dx.$$

Show that T is continuous.

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See Exercises 13 problem 3.