Barycentric calculus
Joensuu
UEF

Eric Lehman

8 février 2017

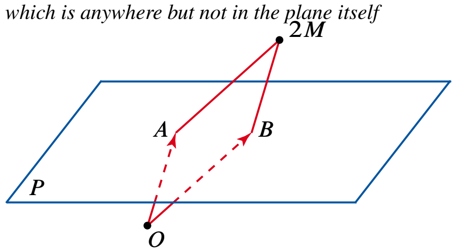
Chapitre 1

Barycenters

To acquire a new concept you have often to give up some prejudices. Here the prejudices are : one cannot multiply a point by a number ; one cannot add two points. The idea of barycentric calculation is to give a meaning to these two operations. Let A and B be two points of a plane P, the symbol $A+B$ indicates an object which does not belong to the plane, but which is associated with a point of P : the point M middle of segment AB. $A+B$ will be the double of the point M :

$$A + B = 2M$$

We recognize the usual vector calculus. Let us invent a point O which is anywhere but not in the plane itself



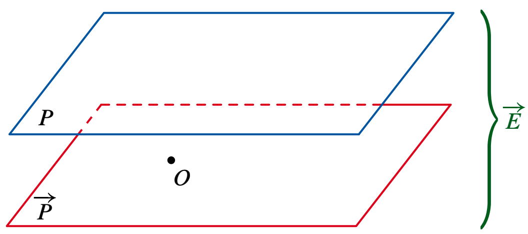
$$\overrightarrow{OA} + \overrightarrow{OB} = 2\overrightarrow{OM}$$

The new object, $2M$, is a VECTOR *of a space $\overrightarrow{E}$ which includes the plane P. We will see that this space $\overrightarrow{E}$ contains also the set $\overrightarrow{P}$ of the vectors $\overrightarrow{AB}$ for $A \in P$ and $B \in P$. Unfortunately, following the tradition we call "mass points" the elements of $\overrightarrow{E}$ even though these objects are not points but vectors. The linear space or vector space $\overrightarrow{E}$ is called the* universal covering space of the plane P.

We say that a line is a geometric object of dimension 1 because one number is enough to locate a point on a line. A plane has dimension 2. Ordinary space has dimension 3. Space-time has dimension 4. We can invent and use spaces of any dimension : we may think of the dimension as the number of independent parameters necessary to characterize an element.

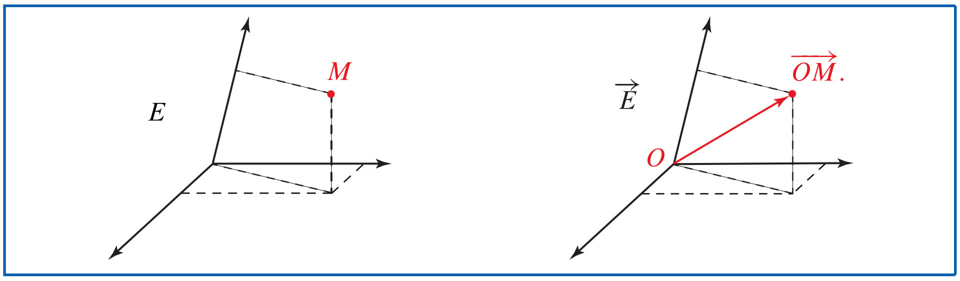
The barycentric calculus in a space of dimension n is actually a vector calculus in a space of dimension $n+1$. Knowledge of linear algebra makes it possible to give simple constructive definitions of affine spaces on any field. They make it possible to better understand the structures underlying geometry, but they are not necessary for a practical use of the barycentric calculus described in this first chapter.

To make it easy we restrict ourselves to barycentric calculus in the real plane. The generalization to an affine space of any dimension and on any field can be done without any difficulty. Let us denote the affine spaces by Latin capitals and the vector spaces by Latin capitals surmounted by an arrow. In an affine line (respectively plane / space) all the points play the same role. In a vectorial line (respectively plane / space) there is one element which is intrinsically different from the others : the zero vector denoted $\vec{0}$, O or simply 0.



§ 1. Observation in (ordinary) space of a horizontal plane of altitude 1

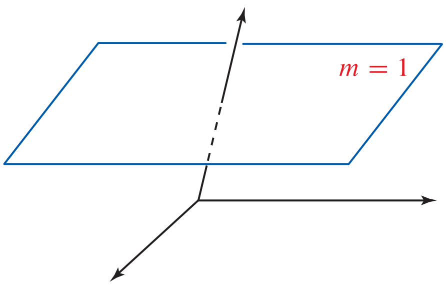
The space E is the "ordinary" space of geometry. We assume that it has a frame $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$. We may choose $\vec{\imath}$ and $\vec{\jmath}$ horizontal and $\vec{k}$ vertical upward. We denote by $\vec{E}$ the set of vectors of E. We suppose O fixed which gives rise to a natural bijection of E onto $\vec{E}$: $M \longmapsto \overrightarrow{OM}$.



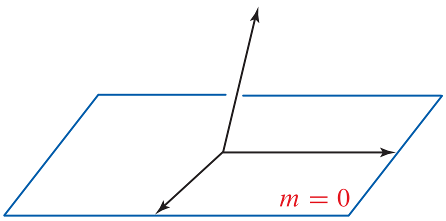
This bijection makes it possible to identify the points of E and the vectors of $\vec{E}$: $M = \overrightarrow{OM}$.
Warning ! This identification has no meaning as long as you do not have chosen a frame or when you have several frames !

Let $(\vec{\imath}, \vec{\jmath}, \vec{k})$ be a basis of $\vec{E}$. There is a natural bijection of $\vec{E}$ onto $\mathbb{R}^3$ which associates to each vector $\overrightarrow{OM}$ its coordinates (x, y, m) such that $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + m\vec{k}$. We denote by m the third coordinate because we will call it "mass" more often than "altitude" .

We want to observe the set P of points M with mass 1 (or altitude 1). Note that P is a plane that does not pass through O. It is called an *affine* plane. The equation of P is $m = 1$.

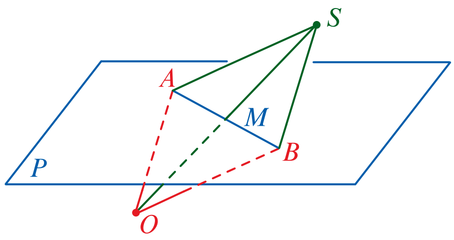


A vector $\overrightarrow{AB}$ in the plane P is a vector belonging to $\vec{E}$ and having its 3rd coordinate null. A point M in E is such that $\overrightarrow{OM} = \overrightarrow{AB}$ where A and B belong to P if and only if M is in the plane of equation $m = 0$. The planes P and $\vec{P}$ are parallel since it is impossible to have at the same time $m = 1$ and $m = 0$.

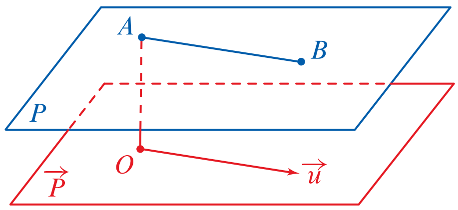


§ 1.. OBSERVATION IN (ORDINARY) SPACE OF A HORIZONTAL PLANE OF ALTITUDE 15

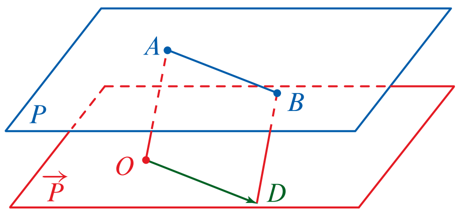
Let A and B be two points in P. We have identified them with the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in $\overrightarrow{E}$. These two vectors can be added. Let S be the point of E such that $\overrightarrow{OS} = \overrightarrow{OA}+\overrightarrow{OB}$. The point S does not belong to P since its mass is $1 + 1 = 2$. But the point M such that $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OS}$ is the middle of the segment AB and belongs to the plane P.



Let A be a point in P and let $\overrightarrow{u} = \overrightarrow{OT}$ be a vector in $\overrightarrow{P}$. We can compute $A + \overrightarrow{u} = \overrightarrow{OA} + \overrightarrow{u}$, getting a vector in $\overrightarrow{E}$ with mass 1 ; this vector is therefore a point B in P. The point B is given by $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{u}$.



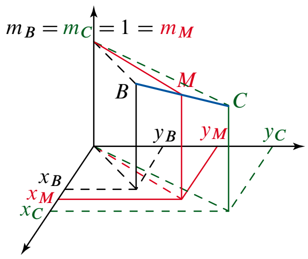
Let A and B be two points in P, identified with the $\overrightarrow{OA}$ and $\overrightarrow{OB}$ vectors in $\overrightarrow{E}$ that can be subtracted. Let D be the point of E such that $\overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$. The point D does not belong to P, but belongs to $\overrightarrow{P}$ since its mass is $1 - 1 = 0$.



We see that the identification of M with $\overrightarrow{OM}$ makes it possible to consider the points in P and the vectors in $\overrightarrow{P}$ as vectors in a common vector space of dimension 3. The idea of barycentric calculus is to consider a plane P and its set of associated vectors $\overrightarrow{P}$ as subsets of a vector space $\overrightarrow{E}$. We can therefore consider objects such as $A + \overrightarrow{u}$ or $A + B$ although we do not put arrows above the letters designating points.

§ 2. Operations on points and vectors of a plane P

Theorem (without proof). Let P be a plane [1] with a frame $(K, \vec{\imath}, \vec{\jmath})$ and let $\overrightarrow{P}$ be the set of vectors of P. The triplet $(\vec{\imath}, \vec{\jmath}, K)$ is a base of a 3-dimensional vector space $\overrightarrow{E}$ containing P and $\overrightarrow{P}$. The points of P are the vectors in $\overrightarrow{E}$ whose 3rd coordinate is 1. The vectors belonging to $\overrightarrow{P}$ are the vectors of $\overrightarrow{E}$ whose 3rd coordinate is 0.

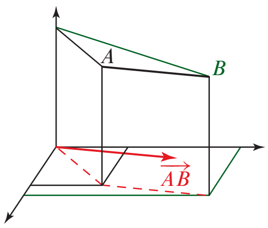


Let M be a vector belonging to $\overrightarrow{E}$ and let (x_M, y_M, m_M) be its coordinates, then $M = x_M\vec{\imath} + y_M\vec{\jmath} + m_M K$. When a vector of $\overrightarrow{E}$ belongs to $\overrightarrow{P}$, it is often written with one or two letters topped by an arrow. For example, $\vec{u} = x_{\vec{u}}\vec{\imath} + y_{\vec{u}}\vec{\jmath} + 0K$. This makes it possible to preserve the usual notations in elementary geometry, but this is not necessary.

Example 1. Middle. Let M be the "middle" (or "midpoint") of the segment BC

$$x_M = \frac{1}{2}(x_B + x_C) \quad y_M = \frac{1}{2}(y_B + y_C) \quad m_M = \frac{1}{2}(m_B + m_C)$$

The relation is written $M = \frac{1}{2}(B + C)$ or $M = \frac{1}{2}B + \frac{1}{2}C$. If B and C belong to P, then M belongs also to P since $\frac{1}{2}(1 + 1) = 1$.



Example 2. Vector belonging to $\overrightarrow{P}$. Let A and B be two points of P, the vector $\overrightarrow{AB}$ is by definition

$$\overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath}$$

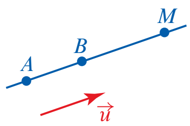
Since $m_A = m_B = 1$, we have $m_A - m_B = 0$ and thus $\boxed{\overrightarrow{AB} = B - A.}$

Example 3. Line in P. Let $A \in P$ and $\vec{u} \in \overrightarrow{P}$. Let $\mathcal{D}$ be the line passing through A and parallel to $\vec{u}$. A point M belongs to this line if and only if there exists a real t such that

$$x_M = x_A + t x_{\vec{u}} \qquad y_M = y_A + t y_{\vec{u}}$$

Since $m_M = m_A = 1$ and $m_{\vec{u}} = 0$, we have also $m_M = m_A + t m_{\vec{u}}$, thus

$$M = A + t\vec{u}$$



Let A and B be two distinct points of P ; a point M belongs to the line AB if there exists t such that $M = A + t(B - A)$ or $M = (1 - t)A + tB$.

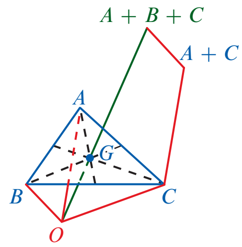
$$\boxed{M \in \text{line } AB \iff \exists t \in \mathbb{R} \quad M = (1-t)A + tB}$$

If a and b are two numbers such that $a + b \neq 0$, we say that M is *barycentre of the points A and B with the masses a and b respectively* if $(a + b)M = aA + bB$. By putting $t = \frac{b}{a+b}$, this relation is equivalent to $M = (1 - t)A + tB$. The line AB is therefore the set of barycenters of A and B.

> A straight line is the set of all the barycenters of any two given points of that line.

1. Strictly speaking, we should specify : affine plane on the field $\mathbb{R}$ or the field $\mathbb{C}$ or any other field K. To allow a simple intuitive image we assume that P is a real affine plane

Example 4. Center of gravity of a triangle of P. Let A, B and C be three points of P. We denote A' the middle of the segment BC, B' the middle of CA and C' the middle of AB. We define the center of gravity G of A, B and C by

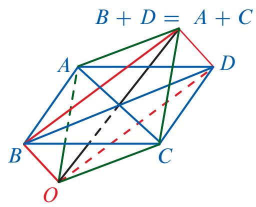


$$3G = A + B + C$$

Or $G = \frac{1}{3}(A + B + C)$. From $2A' = B + C$, we deduce $3G = A + 2A'$. The point G is therefore a barycenter of A and A' and thus belongs to the median AA'. The point G also belongs to the medians BB' and CC'. These three lines are therefore concurrent. The point G is the center of gravity of the triangle ABC. In addition, $3(GA) = 2(A' - A)$ or $\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AA'}$ which is translated as : "center of gravity of a triangle is situated at two-thirds of the medians".

Exercice 1. Show that G is also the center of gravity of the triangle $A'B'C'$.

Example 5. Parallelogram of P. Let A, B, C and D four points of P. The quadrilateral $ABCD$ is a parallelogram if one of the following equivalent conditions is satisfied



(i) $B - A = C - D$ (translation : $\overrightarrow{AB} = \overrightarrow{DC}$)

(ii) $B + D = A + C$ (translation : diagonals intersect each other in their midpoints)

(iii) $D - A = C - B$ (translation :$\overrightarrow{AD} = \overrightarrow{BC}$)

(iv) $\exists\alpha : B - A = \alpha(C - D)$ et $\exists\beta : D - A = \beta(C - B)$ (translation : $AB \parallel DC$ and $AD \parallel BC$)

(Condition (iv) is equivalent only if the points are not aligned.) The only non-trivial verification is the verification of (iv) $\to$ (i) : the data relations express the existence of two real numbers α and β such that

$$\begin{cases} \overrightarrow{AB} - \alpha\overrightarrow{AC} + \alpha\overrightarrow{AD} = 0 \\ -\beta\overrightarrow{AB} + \beta\overrightarrow{AC} - \overrightarrow{AD} = 0 \end{cases}$$

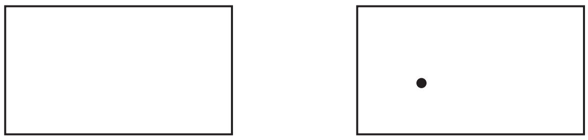
Hence $(1 - \alpha\beta)\overrightarrow{AB} + (\alpha\beta - \alpha)\overrightarrow{AC} = 0$. Since A, B and C are not aligned, the $(1 - \alpha\beta)$ and $(\alpha\beta - \alpha)$ coefficients are both null and $\alpha = \alpha\beta = 1$.

Exercice 2. Show that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

affine $\nleftrightarrow$ vector affine $\nleftrightarrow$ metric

The word "affine" is opposite either to the word "vectorial" or to the word "metric".

In the first meaning, the affine plane P is the plane whose elements are points as opposed to the vectorial plane $\overrightarrow{P}$ whose elements are vectors.



In an affine plane all points play the same role. In a vectorial plane, there is a particular element : the vector $\vec{0}$.

The set of vectors of an affine plane P form the associated vectorial plane denoted by $\overrightarrow{P}$. If one chooses a point A in an affine plane, it becomes "vectorialised" and there is a natural bijection between the points of P and the vectors of $\overrightarrow{P}$:

$$P \longrightarrow \overrightarrow{P}, \quad B \longmapsto \overrightarrow{AB}$$

Conversely, if one pretends to forget where the null vector is, one can always consider the vectorial plane as an affine plane. The vectorial plane associated with $\overrightarrow{P}$ considered as an affine plane is the vectorial plane $\overrightarrow{P}$ itself. This is why, when we compute the derivative of a moving point $M(t)$ in an affine plane, the derivative is not a point but a vector, the velocity vector, and when the velocity vector is derived, the derivative is again a vector, the vector acceleration.

The second meaning of the word "affine" is opposite to the notion of "metric" : a property is affine if it is independent of the choice of a scalar product. The properties or concepts that presuppose the choice of a scalar product are called metric.

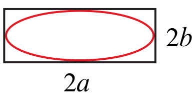
Some affine properties and affine notions :

- point alignment
- parallelism
- middle and more generally barycenters
- ellipse
- area ratio, algebraic area ratio

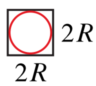
Some properties or notions which are not affines :

- scalar product (inner product)
- orthogonality and perpendicularity
- distances
- angles
- circle
- areas, algebraic areas

APPLICATION. The ratio of the area of an ellipse to the circumscribed rectangle is equal to the ratio of the area of the disc to a circumscribed square, since an affinity transforms one of the figures into the other :



$$\frac{\text{area of the ellipse}}{2a \times 2b} = \frac{\text{area of the disc}}{(2R)^2}$$



Areas are metric notions, but the ratios of areas are affine. One knows how to compute the areas of the square, the disc and the rectangle. From this we deduce the metric formula giving the area of the ellipse as a function of the semi-major axis a and the semi-minor axis b :

$$\pi ab$$

1. The two different meanings of the word "affine".

§ 3. Barycentric calculus in a real affine plane

Definition. Let P be a real affine plane. A *mass point* of P is a couple mA where $m \in \mathbb{R}^*$ and $A \in P$. Let $(m_i A_i)_{i=1,2,\dots,n}$ be a finite sequence of mass points of P such that $\sum_{i=1}^n m_i \neq 0$. We call *barycenter of the points A_i with the masses m_i* the point G such that $\left(\sum_{i=1}^n m_i\right) G = \sum_{i=1}^n m_i A_i$.

Proposition. Let P be a real affine plane and $(m_i A_i)_{i=1,2,\dots,n}$ a finite sequence of mass points of P such that $\sum_{i=1}^n m_i \neq 0$. The barycenter of the points A_i assigned to the masses m_i is the unique point G such that

$$\sum_{i=1}^n m_i \overrightarrow{GA_i} = \vec{0} \qquad (*)$$

or, equivalently, such that for a point Ω in P, we have

$$\overrightarrow{\Omega G} = \frac{1}{\sum_{i=1}^n m_i} \sum_{i=1}^n m_i \overrightarrow{\Omega A_i} \qquad (**)$$

(If the relation $(**)$ is true for one point Ω, then it is true for all other choices. If we choose $\Omega = G$ then we get $(*)$).

C 1. Prove the proposition above.

Remark. if $\sum_{i=1}^n m_i = 0$, the element $\sum_{i=1}^n m_i A_i$ in $\overrightarrow{E}$ belongs to $\overrightarrow{P}$. For any point K in the plane P, we have :

$$\sum_{i=1}^n m_i A_i = \sum_{i=1}^n m_i A_i - 0K = \sum_{i=1}^n m_i A_i - \sum_{i=1}^n m_i K = \sum_{i=1}^n m_i (A_i - K) = \sum_{i=1}^n m_i \overrightarrow{KA_i}$$

Associativity of barycenters

The "associativity of barycenters" is simply the associativity of the addition in the linear space $\overrightarrow{E}$. It may be written :

$$\begin{aligned}(m_1 + \cdots + m_n)G &= m_1 A_1 + \cdots + m_n A_n \\ &= (m_1 A_1 + \cdots + m_p A_p) + (m_{p+1} A_{p+1} + \cdots + m_n A_n) \\ &= (m_1 + \cdots + m_p)G_1 + (m_{p+1} + \cdots + m_n)G_2\end{aligned}$$

There are many awkward ways to express that property. We give here some usual formulation just to show how awful it looks.

Proposition. Let $(m_i A_i)_{i=1,2,\dots,n}$ be a finite sequence of mass points in P such that $\sum_{i=1}^{n} m_i \neq 0$ and let G be the barycenter of the points $A_1, A_2, \dots, A_n$ assigned to the masses $m_1, m_2, \dots, m_n$.
We suppose that $\sum_{i=1}^{p} m_i \neq 0$ and $\sum_{i=p+1}^{n} m_i \neq 0$.
Let us denote by G_1 the barycenter of the points $A_1, A_2, \dots, A_p$ assigned to the masses m_1, $m_2, \dots, m_p$ and let G_2 be the barycenter of the points $A_{p+1}, A_{p+2}, \dots, A_n$ assigned to the masses $m_{p+1}, m_{p+2}, \dots, m_n$. The point G is then the barycenter of G_1 and G_2 assigned the masses $(m_1 + m_2 + \cdots + m_p)$ and $(m_{p+1} + m_{p+2} + \cdots + m_n)$. Forget it as soon as possible ! ! !

Theorem of parallel lines (in french Thalès' theorem)

Proposition. Let P be a real affine plane and let $(m_i A_i)_{i=1,2,\dots,n}$ be a sequence of n mass points in P such that $\sum_{i=1}^{n} m_i \neq 0$. If all the points A_i are on a same line $\mathcal{D}$, then their barycenter G belongs to that line $\mathcal{D}$.

Proof. Let $\overrightarrow{u}$ be a vector in $\overrightarrow{\mathcal{D}}$ different from $\overrightarrow{0}$. In the relation $(**)$ above, let us choose a point Ω belonging to $\mathcal{D}$; all the vectors $\overrightarrow{\Omega A_i}$ are linearly dependent with $\overrightarrow{u}$, and thus $\overrightarrow{\Omega G}$ is also linearly dependent with $\overrightarrow{u}$, which proves that $G \in \mathcal{D}$. □

Lemma. Let A, B and C be three aligned points such that $A \neq C$ and let $\lambda \in \mathbb{R}$; then

$$\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \lambda \iff B \text{ is barycenter of } A \text{ and } C \text{ assigned to masses } 1-\lambda \text{ and } \lambda$$

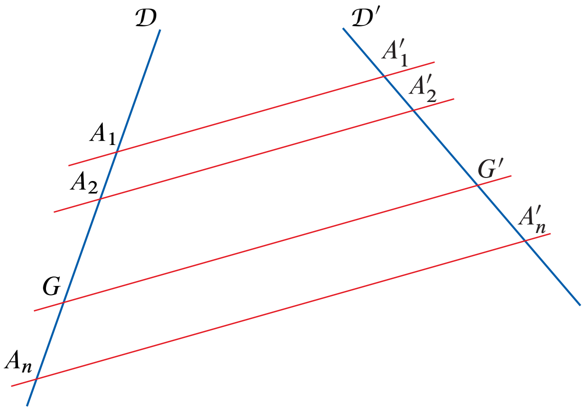
Proof. The relation $\frac{\overrightarrow{AB}}{\overrightarrow{AC}} = \lambda$ means $\overrightarrow{AB} = \lambda \overrightarrow{AC}$, that is to say $B - A = \lambda(C - A)$, and that relation is equivalent to

$$B = (1-\lambda)A + \lambda C \qquad \square$$

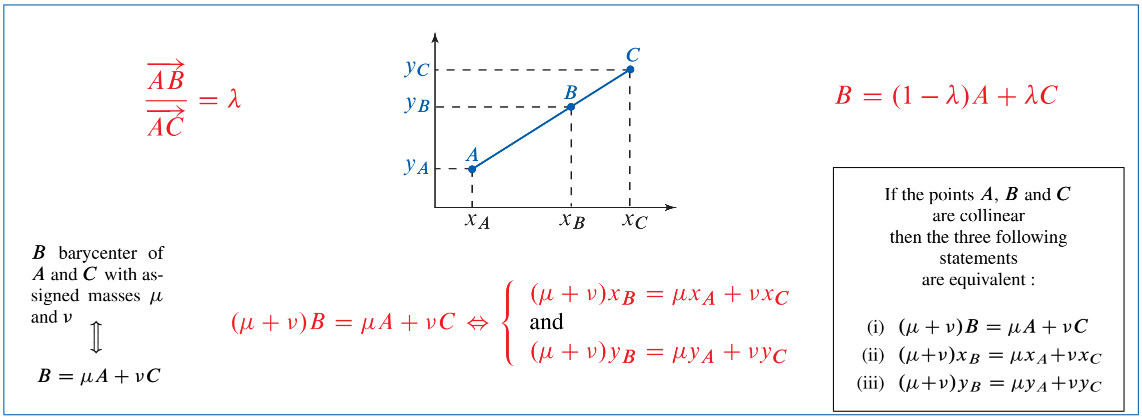
Proposition and definition. The parallelism is an equivalence relation in the set of lines in P. The equivalence classes are called line directions. (The direction of a line δ is thus the set of lines parallel to δ).

Theorem. The parallel projection of a line on an other preserves the barycenters.

Proof. Let $\mathcal{D}$ and $\mathcal{D}'$ be two lines in P and let δ be a line direction which does not contain $\mathcal{D}$ nor $\mathcal{D}'$. Let $(m_i A_i)_{i=1,2,\dots,n}$ be a finite sequence of mass points of P such that $\sum_{i=1}^{n} m_i \neq 0$ and such that the points A_i all belong to the line $\mathcal{D}$. For each i, let A'_i be the intersection of $\mathcal{D}'$ with the line belonging to the direction δ and containing A_i and let G' be the intersection of $\mathcal{D}'$ with the line belonging to the direction δ and containing G. We have to prove that if G is the barycenter of the A_i assigned with the masses m_i, then G' is the barycenter of the points A'_i assigned with the same masses m_i.



If $n = 2$, it follows immediately the theorem of parallel lines and the lemma above. To prove the theorem for any n, use mathematical induction and the associativity of barycenters. □
Remark. This theorem is just a special case of the general theorem that says that the maps that preserve barycenters are the affine maps.



2. A sketch that one encounters often

§ 4. Barycentric coordinates

In the following paragraph we use determinants of order 3.

Let us change the basis of the linear space $\overrightarrow{E}$. The new basis is a triplet of vectors which are three non collinear points in P.

Définition. Let ABC be a triangle in P (the three vertices are not collinear). The *normalized barycentric coordinates of a point* M are the coordinates α, β and γ of M as a vector in the basis (A, B, C) of the linear space $\overrightarrow{E}$

$$M = \alpha A + \beta B + \gamma C$$

Every triplet $(\lambda\alpha, \lambda\beta, \lambda\gamma)$ which is a non-zero multiple of (α, β, γ) is a triplet of *unnormalized coordinates of the same* M. In fact, they are the coordinates of λM in the basis (A, B, C).

Remark. Let ABC be a (non-flat) triangle in P. Let U and V be two points with normalized barycentric coordinates $(\alpha_U, \beta_U, \gamma_U)$ and $(\alpha_V, \beta_V, \gamma_V)$ respectively. A point M in P is barycenter of U and V assigned with the masses λ and μ if and only if the coordinates $(\alpha_M, \beta_M, \gamma_M)$ of M are such that α_M is barycenter of α_U and α_V assigned with masses λ and μ, β_M is barycenter of β_U and β_V assigned with masses λ and μ and γ_M is barycenter of γ_U and γ_V assigned with masses λ and μ.

Exercice 3. Prove the statement done in the above remark.

Equation of a line. Let ABC be a triangle in P. We consider the normalized barycentric coordinates relative to the triangle ABC. Three points M, U and V in P are collinear if and only if they are as vectors in $\overrightarrow{E}$ linearly dependent, that is to say that their determinant is zero

$$\begin{vmatrix} \alpha_M & \alpha_U & \alpha_V \\ \beta_M & \beta_U & \beta_V \\ \gamma_M & \gamma_U & \gamma_V \end{vmatrix} = 0$$

Exercice 4. Prove that the equality above is a necessary and suffisent condition for collinearity even if the coordinates are not normalized.

We suppose $U \neq V$ which means that the second and the third columns of the determinant are linearly independent. Then a point M with barycentric coordinates (normalized or not) (α, β, γ) belong to the line UV if and only if its coordinates are such that

$$p\alpha + q\beta + r\gamma = 0$$

where

$$\begin{cases} p &= \lambda\,(\beta_U\gamma_V - \beta_V\gamma_U) \\ q &= \lambda\,(\gamma_U\alpha_V - \gamma_V\alpha_U) \\ r &= \lambda\,(\alpha_U\beta_V - \alpha_V\beta_U) \end{cases}$$

This relation between α, β and γ is called the *equation of the line* UV.

Remark. Here (p, q, r) is a solution of the system

$$\begin{cases} \alpha_U\, p + \beta_U\, q + \gamma_U\, r = 0 \\ \alpha_V\, p + \beta_V\, q + \gamma_V\, r = 0 \end{cases}$$

It is often useful to know how to solve a linear system of 2 equations with 3 unknown when the rank of the system is 2

$$\begin{cases} ax + by + cz = 0 \\ a'x + b'y + c'z = 0 \end{cases}$$

The solution is the set of triplets (x, y, z) which are multiples of

$$\begin{cases} x_0 = bc' - cb' \\ y_0 = ca' - ac' \\ z_0 = ab' - ba' \end{cases}$$

To remember this result, one may think of the vectorial product in $\mathbb{R}^3$

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$$

Intersection of two lines. To find the coordinates of the point of intersection of two distinct lines we have to solve the linear system of rank 2

$$\begin{cases} p\,\alpha + q\,\beta + r\,\gamma = 0 \\ p'\,\alpha + q'\,\beta + r'\,\gamma = 0 \end{cases}$$

As we have just seen above the solutions of this system are the multiples of

$$\begin{cases} \alpha_0 = qr' - rq' \\ \beta_0 = rp' - pr' \\ \gamma_0 = pq' - qp' \end{cases}$$

Notice the we cannot normalize these coordinates if (and only if) $\alpha_0 + \beta_0 + \gamma_0 = 0$. This relation expresses then the parallelism of the two lines. From that we deduce the following proposition.

Proposition. Two lines whose equations in barycentric coordinates

$$p\,\alpha + q\,\beta + r\,\gamma = 0$$

and

$$p'\,\alpha + q'\,\beta + r'\,\gamma = 0$$

are parallel if and only if

$$qr' - rq' + rp' - pr' + pq' - qp' = 0$$

Exemple. Remember that the barycentric coordinates of B and C relative to the triangle ABC are respectively $(0, 1, 0)$ and $(0, 0, 1)$. The equation of the line BC is thus

$$\begin{vmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \gamma & 0 & 1 \end{vmatrix} = 0$$

or simpler

$$\alpha = 0$$

A line with equation $p\,\alpha + q\,\beta + r\,\gamma = 0$ is parallel to BC if and only if

$$q\,0 - r\,0 + r\,1 - p\,0 + p\,0 - q\,1 = 0$$

or simply $q = r$. Example : the line ℓ which goes through the midpoints C' and B' of the sides AB and AC. The equation of the line ℓ is

$$\begin{vmatrix} \alpha & 1 & 1 \\ \beta & 1 & 0 \\ \gamma & 0 & 1 \end{vmatrix} = \alpha - \beta - \gamma = 0$$

The line ℓ is parallel to BC since its coefficients (p, q, r) are equal to $(1, -1, -1)$ and we have indeed $q = r$.

Barycentric coordinates in terms of algebraic areas. From now on, we suppose that a scalar product (or inner product) has been defined on $\overrightarrow{P}$. Then the metric notions are defined on P. Examples of such notions are : distances, angles, heights and bisectors in a triangle (the medians are just affine notions and need no scalar product to be defined). We suppose also that the distance between P and $\overrightarrow{P}$ has been chosen. Let us take it equal to 1.

Recall that the algebraic volume of a parallelepiped constructed on 3 linearly independent vectors is equal to the determinant of these 3 vectors. Recall that the algebraic area of a parallelogram constructed on two independent vectors is equal to the determinant of these two vectors. Recall also the formula giving the volume of a tetrahedron

$$V = \frac{1}{3}\text{height} \times \text{base} = \frac{1}{3}\text{heigth} \times \text{algebraic area of the triangle at the bottom}$$

Using the properties of determinants like $\det(\alpha A + \beta B + \gamma C, B, C) = \alpha \det(A, B, C)$, and so on, we see that the following sequences are proportional :

$$\begin{array}{cccc} \det(A,B,C) & \det(M,B,C) & \det(M,C,A) & \det(M,A,B) \\ \text{area } ABC & \text{area } MBC & \text{area } MCA & \text{area } MAB \\ 1 & \alpha & \beta & \gamma \end{array}$$

An immediate consequence of this is that the barycentric coordinates of the center of the incircle ABC are (a, b, c) where $a = BC$, $b = CA$ and $c = AB$ are the lengths of the sides of the triangle ABC. In the same way we get that the barycentric coordinates of inexcircles are $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$.

Distance between two points. Let U and V be two points in the plane P where the barycentric coordinates are relative to the triangle ABC. We put $a = BC$, $b = CA$ and $c = AB$. Let $(\alpha_U, \beta_U, \gamma_U)$ be barycentric coordinates of U and $(\alpha_V, \beta_V, \gamma_V)$ barycentric coordinates of V. We put $m_U = \alpha_U + \beta_U + \gamma_U$ and $m_V = \alpha_V + \beta_V + \gamma_V$. The square of the distance between the points U and V is given by

$$UV^2 = \frac{\displaystyle\sum_{\text{perm circ}} \Big[-a^2 \big(m_V \beta_U - m_U \beta_V\big)\big(m_V \gamma_U - m_U \gamma_V\big)\Big]}{m_U^2 m_V^2}$$

where the sum has to be taken over the circular permutions $(a, \alpha_U, \alpha_V) \to (b, \beta_U, \beta_V) \to (c, \gamma_U, \gamma_V) \to (a, \alpha_U, \alpha_V)$. The first term of that sum is the expression between brackets, the second term is obtained by replacing a by b, β_V by γ_V, and so on. The third and last term begins as $\Big[-c^2 \big((\alpha_U + \beta_U + \gamma_U)\alpha_V - \cdots$

If we use normalized barycentric coordinates, the formula takes the simpler shape

$$UV^2 = -a^2(\beta_U - \beta_V)(\gamma_U - \gamma_V) - b^2(\gamma_U - \gamma_V)(\alpha_U - \alpha_V) - c^2(\alpha_U - \alpha_V)(\beta_U - \beta_V)$$

To prove these two formulae, let us first notice that the first formula follows the simpler one by replacing in it α_U by $\frac{\alpha_U}{\alpha_U + \beta_U + \gamma_U}, \ldots, \gamma_V$ by $\frac{\gamma_V}{\alpha_V + \beta_V + \gamma_V}$.

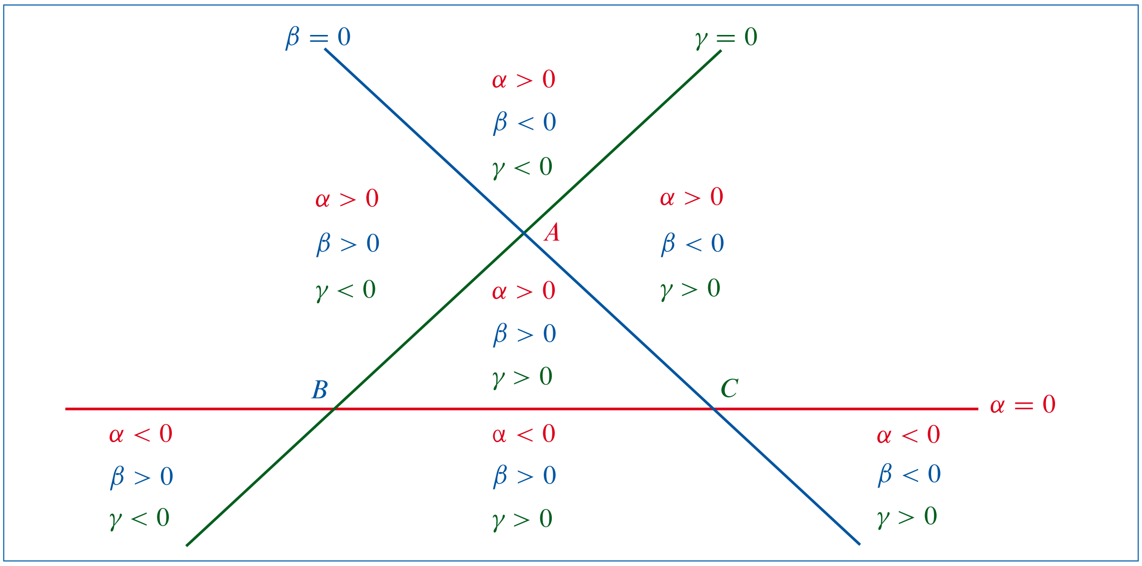
On the other hand, if we make the computations with normalized coordinates

$$\overrightarrow{UV} = V - U = (\alpha_V - \alpha_U)A + (\beta_V - \beta_U)B + (\gamma_V - \gamma_U)C = (\beta_V - \beta_U)\overrightarrow{AB} + (\gamma_V - \gamma_U)\overrightarrow{AC}$$

Then

$$UV^2 = (\beta_U - \beta_V)^2 c^2 + (\gamma_U - \gamma_V)^2 b^2 + 2(\beta_U - \beta_V)(\gamma_U - \gamma_V)\overrightarrow{AB} \cdot \overrightarrow{AC}$$

Since $2\overrightarrow{AB} \cdot \overrightarrow{AC} = b^2 + c^2 - a^2$, the coefficient of a^2 is the one expected. By symmetry the coefficients of b^2 and of c^2 are also those of the formula. Now, if you are not convinced by these arguments, make the computations to the end using the relations $\alpha_U + \beta_U + \gamma_U = 1$ and $\alpha_V + \beta_V + \gamma_V = 1$.



3. Regionalization of the plane following normalized barycentric coordinates : $\alpha + \beta + \gamma = 1$.

Exercice 5. Let $ABCD$ be a square in a usual euclidean plane. Find the normalized barycentric coordinates of the point D relative to the triangle ABC.

Exercice 6. Let $ABCD$ be a parallelogram in a usual affine plane. Find the normalized barycentric coordinates of the point D relative to the triangle ABC. Compare with exercise 5.

Exercice 7. Let A, B, C and D be points of an affine plane P. When is it possible to define $\frac{\overrightarrow{AB}}{\overrightarrow{CD}}$? when it is, what is the meaning you give to that symbol ?

Exercice 8. Let ABC be a triangle in an affine plane P. Let d_1, d_2 and d_3 be three lines in P. We write the equations of these three lines

$$p_i\alpha + q_i\beta + r_i\gamma = 0$$

for $i = 1$, 2 and 3. Show that these three lines are concurrent (that means that they have a common point) if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

Exercice 9. Write the barycentric equations of the three medians of a triangle (barycentric coordinates relatively to that triangle). Check that these three lines are concurrent in a point G. What are the coordinates of G ?

** **Exercice 10.** Let ABC be a triangle in an euclidean plane P. We denote the length of the sides $a = BC$, $b = CA$ and $c = AB$.

a) Show that the scalar product $\overrightarrow{AB} \cdot \overrightarrow{AC} = \frac{1}{2}(-a^2 + b^2 + c^2)$.

b) Show that $\Big((a^2 - b^2 + c^2)(a^2 + b^2 - c^2), (a^2 + b^2 - c^2)(-a^2 + b^2 + c^2), (-a^2 + b^2 + c^2)(a^2 - b^2 + c^2) \Big)$ are barycentric coordinates of the orthocenter of the triangle ABC.

Exercice 11. Let ABC be a triangle in an affine plane P. Let R be a point on the line BC, S a point on CA and T a point on AB. a) Prove that if a point R is such that $\frac{\overrightarrow{RB}}{\overrightarrow{RC}} = k$ where k is a real number, then the barycentric equation of the line AR is $k\beta - \gamma = 0$.

b) Prove Ceva's theorem : the lines AR, BS and CT are concurrent if and only if

$$\frac{\overrightarrow{RB}}{\overrightarrow{RC}} \frac{\overrightarrow{SC}}{\overrightarrow{SA}} \frac{\overrightarrow{TC}}{\overrightarrow{TB}} = -1$$

c) Prove Menelaus' theorem : the points R, S and T are collinear if and only if

$$\frac{\overrightarrow{RB}}{\overrightarrow{RC}} \frac{\overrightarrow{SC}}{\overrightarrow{SA}} \frac{\overrightarrow{TC}}{\overrightarrow{TB}} = 1$$

*** **Exercice 12.** Feuerbach's theorem, published by Feuerbach in 1822, states more generally that the nine-point circle is tangent to the three excircles of the triangle as well as its incircle.

Prove Feuerbach's theorem using barycentric calculus.