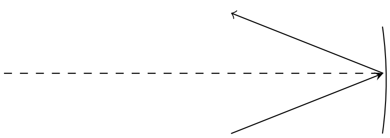
Chapitre 1

One-dimensional mechanics on a half-line of two mass-points with equal masses

Mechanics is one of the motivations to study billiards. Her we see how to translate into geometry the simplest mechanical system.

§ 1. What is a mathematical billiard ?

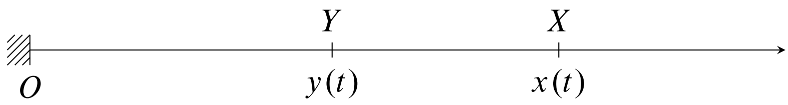
We simplify as much as possible : there will be only ONE ball and this ball will be reduced to one point. We accept billiard tables of any shape in any kind of geometry. Let us call D the billiard table. To begin, we shall suppose D in the usual Euclidean plane and we denote the border of D by γ. The "ball" is moving freely inside D, that is, with a constant speed inside D but when it comes to the border it bounces following usual reflection laws : it comes off with a vectorial speed symmetric to the ingoing vectorial speed with respect to the normal to the border



Since there is no friction, the travel of the point will never end. The set of successive locations of the point is its trajectory. We want to study the aspects of that trajectory when time goes on forever.

§ 2. Two equal mass-points on a half-line

Let us start with an example : we consider two point-masses X and Y on a half-line. We put the origin of the half-line O at its end-point and call the abscissa of X by $x(t)$ and the abscissa of Y by $y(t)$ depending on the time t. We suppose that they are placed on the half-line in such a way that $y(0) \leqslant x(0)$



To make it even simpler we suppose that X and Y have the same mass m and that the shock between them is elastic, which means in this case that the two point-masses just exchange their speeds. Thus we have $y(t) \leqslant x(t)$ for all t. What happens if Y comes to the end-point O : it just bumps back with the same speed as it has arriving at the point O.

Let us call v_x the speed of the mass-point X before a shock and u_x the speed of that point after the shock. In the same way, let v_y and u_y be the speeds of Y before and after the shock :

1. Shock between X and Y :

$$\begin{cases} u_x = v_y \\ u_y = v_x \end{cases}$$

2. Shock between Y and O :

$$\begin{cases} u_x = v_x \\ u_y = -v_y \end{cases}$$

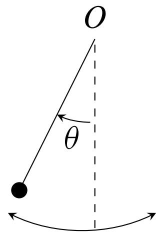
§ 3. Configuration space of a mechanical system

We want to represent the state of the mechanical system by one point. All the possible points form the configuration space. This is not a mathematical definition, but let us illustrate the concept by some examples.

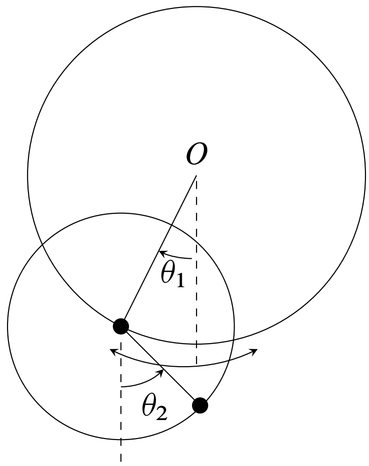
1. Example 1. A pendulum can be described by a moving segment ℓ with one end fixed in a point O and moving freely in a vertical plane. Let θ be the oriented angle from the vertical line pointing down and the segment ℓ. If the pendulum oscillates between two extremal angles,

we can take as configuration space the set $[-\theta_{\max}, \theta_{\max}]$; this set is just a closed finite interval of $\mathbb{R}$.

If we manage so that the pendulum may make a complete tour, we have to take $[-\pi, \pi]$, but the two ends of that interval describe the same configuration : we have to glue them together getting a circle. A nice configuration space will be the unit circle in $\mathbb{R}^2$, which is denoted S^1 since it is a 1-dimensional sphere.



2. Example 2. A double pendulum organized in such a way that both pendula are in the same plane and can go around a whole circle.



$S^1 \times S^1 = T^2$, two-dimensional torus.

3. Example 3. An oriented line going through the origin in $\mathbb{R}^3$. Configuration space $= S^2$, ordinary sphere.
4. Example 4. A non-oriented line in the plane. Configuration space $=$ the projective plane $P\mathbb{R}^2$.
5. Example 5. A non-oriented line in the plane. Configuration space $=$ Möbius strip.

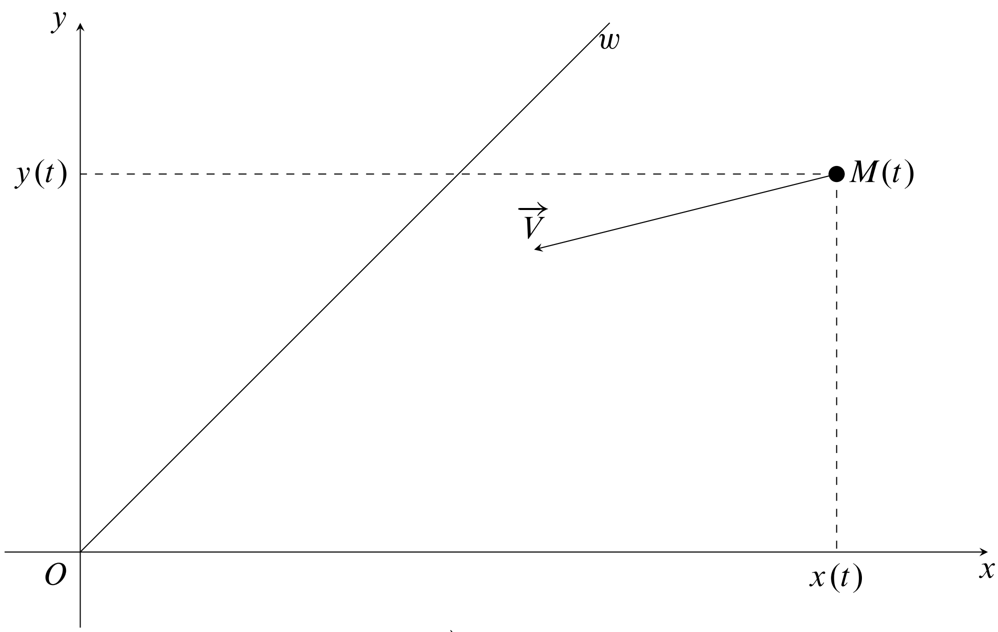
§ 4. A configuration space of the system of §2

Question : give a configuration space for the two mass-points X and Y on a half-line. The abscissae of these points are depending on the time t. Let us call them $x(t)$ and $y(t)$. The constraints are

$$0 \leqslant y(t) \leqslant x(t)$$

Thus we can choose as configuration space the wedge in $\mathbb{R}^2$ limited by the lines with equations $y = 0$ (the x-axis) and $x = y$ (the first bisector). Let us use a metric such that the canonical basis $((1,0),(0,1))$ is orthonormal. Then the angle of the wedge is 45° or better $\frac{\pi}{4}$ rad. The mechanical system is described by the point $M(t) = (x(t), y(t))$ in the wedge. The velocity of M is a vector

$$\overrightarrow{V} = \frac{\mathrm{d}M}{\mathrm{d}t} = (x'(t), y'(t))$$

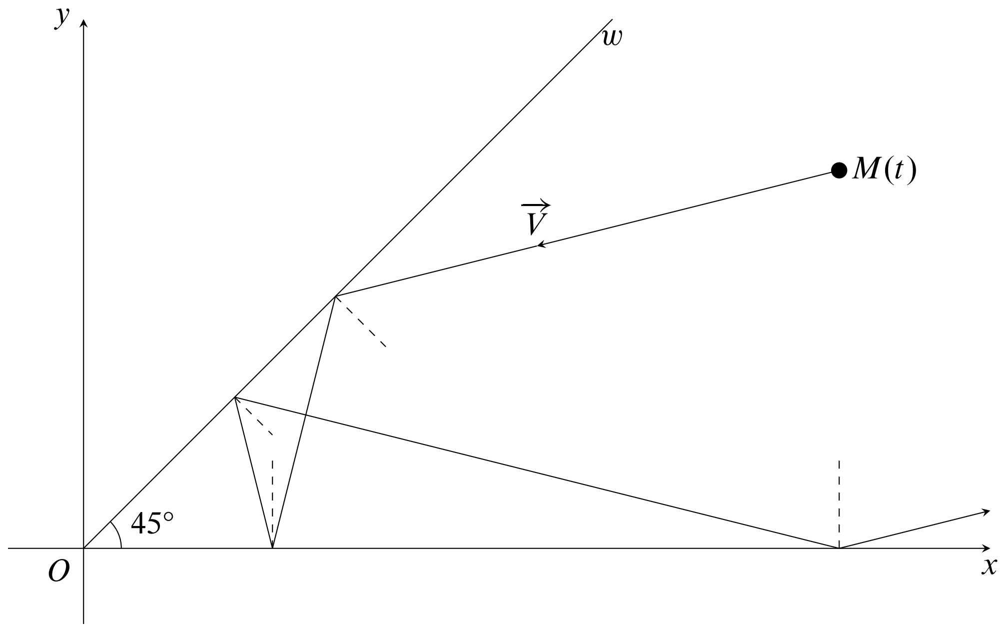


When there is a shock let us call $\overrightarrow{V}$ the velocity of M before the shock and $\overrightarrow{U}$ after. If the shock is between the two mass-points X and Y we have seen that $\overrightarrow{V} = (v_x, v_y)$ is changed into $\overrightarrow{U} = (v_y, v_x)$, that is that $\overrightarrow{V}$ and $\overrightarrow{U}$ are symmetrical relatively to the first diagonal. If the shock is between Y and the origin $\overrightarrow{V} = (v_x, v_y)$ is changed into $\overrightarrow{U} = (v_x, -v_y)$, thus $\overrightarrow{U}$ is the image of $\overrightarrow{V}$ in the symmetry with respect to the x-axis.

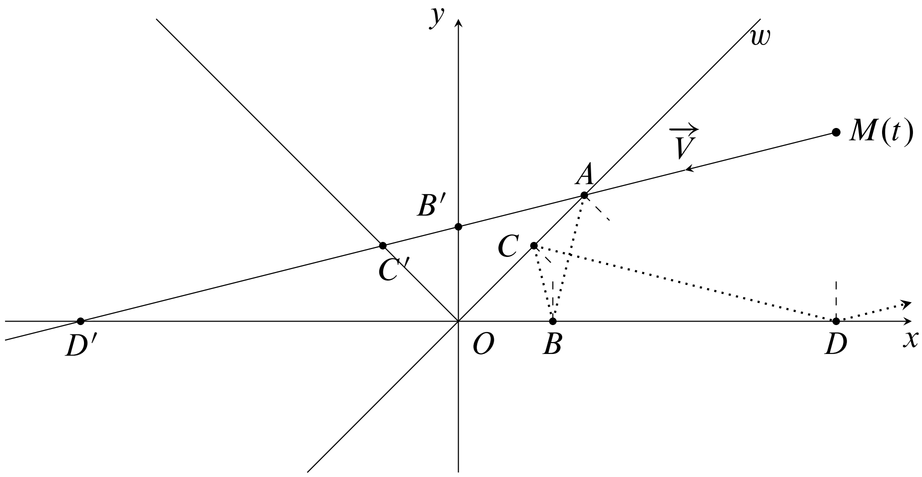
Finally we see that we describe the mechanical system by a billiard whose table is the wedge xOw where Ow is the ray such that the oriented angle (Ox, Ow) has mesure 45° or $\frac{\pi}{4}$.

§ 5. Study of a trajectory in the configuration space of the system of §2

By applying the law of reflection we get



But instead of reflecting the trajectory we may take the symmetric of the domain D and then through the image of Ox in that symmetry and so on. This method of "unfolding" is shown in the following picture.


y
w
M(t)
V
A
B′
C
C′
D′
O
B
D
x