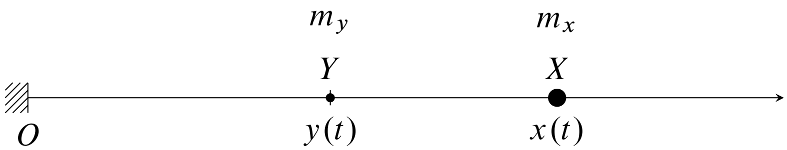
Chapitre 2

One-dimensional mechanics on a half-line of two mass-points with any masses

§ 1. Solving the conservation laws equations

We consider again two point-masses X and Y on a half-line, but this time with masses m_x and m_y that might be different. We put the origin of the half-line O at its end-point and call the abscissa of X by $x(t)$ and the abscissa of Y by $y(t)$ depending on the time t. We still suppose that they are placed on the half-line in such a way that $y(0) \leqslant x(0)$



Let us call v_x the speed of the mass-point X before a shock and u_x the speed of that point after the shock. In the same way, let v_y and u_y be the speeds of Y before and after the shock :

1. Shock between X and Y : we have to solve the system

$$\begin{cases} m_x u_x + m_y u_y = m_x v_x + m_y v_y \\ \frac{1}{2} m_x u_x^2 + \frac{1}{2} m_y u_y^2 = \frac{1}{2} m_x v_x^2 + \frac{1}{2} m_y v_y^2 \end{cases}$$

We introduce new variables $\overline{x} = \sqrt{m_x}x$ and $\overline{y} = \sqrt{m_y}y$ and the corresponding derivatives $\overline{v_x} = \sqrt{m_x}v_x$, $\overline{u_x} = \sqrt{m_x}u_x$, $\overline{v_y} = \sqrt{m_y}v_y$ and $\overline{u_y} = \sqrt{m_y}u_y$. The system becomes

$$\begin{cases} \sqrt{m_x}\ \ \overline{u_x} + \sqrt{m_y}\ \ \overline{u_y} = \sqrt{m_x}\ \ \overline{v_x} + \sqrt{m_y}\ \ \overline{v_y} \\ \overline{u_x}^2 + \overline{u_y}^2 = \overline{v_x}^2 + \overline{v_y}^2 \end{cases}$$

2. Shock between Y and O : as before

$$\begin{cases} u_x = v_x \\ u_y = -v_y \end{cases} \text{and thus} \begin{cases} \overline{u_x} = \overline{v_x} \\ \overline{u_y} = -\overline{v_y} \end{cases}$$

§ 2. The configuration space

We choose a subset of the real plane with coordinates $\overline{x}$ and $\overline{y}$. The inequalities $0 \leqslant y(t) \leqslant x(t)$ become $0 \leqslant \overline{y}(t)$ and $\frac{\overline{y}(t)}{m_y} \leqslant \frac{\overline{x}(t)}{m_x}$. The domain D is now the wedge between the positive part of the $\overline{x}$-axis and the ray Ow in the first quadrant with equation $\overline{y} = \frac{\sqrt{m_y}}{\sqrt{m_x}}\overline{x}$ or

$$\overline{y} = \sqrt{\frac{m_y}{m_x}}\ \overline{x}$$

This is a line with slope $\sqrt{\frac{m_y}{m_x}}$, thus the angle of our wedge is

$$\alpha = \arctan \sqrt{\frac{m_y}{m_x}}$$

But why did we choose this domain D instead of the one we had before ? Tt is to get a billiard to describe the evolution of the system. Let us prove that :

Introduce the vectors $\overrightarrow{V} = (\overline{v_x}, \overline{v_y})$ and $\overrightarrow{U} = (\overline{u_x}, \overline{u_y})$. The equations for a shock can be written in the following way :

1. Shock between X and Y :

$$\begin{cases} (\sqrt{m_x}, \sqrt{m_y}) \cdot (\overrightarrow{U} - \overrightarrow{V}) = 0 \\ ||\overrightarrow{U}||^2 = ||\overrightarrow{V}||^2 \end{cases}$$

Thus $\overrightarrow{U}$ and $\overrightarrow{V}$ are symmetrical with respect to the ray Ow.

2. Shock between Y and O :

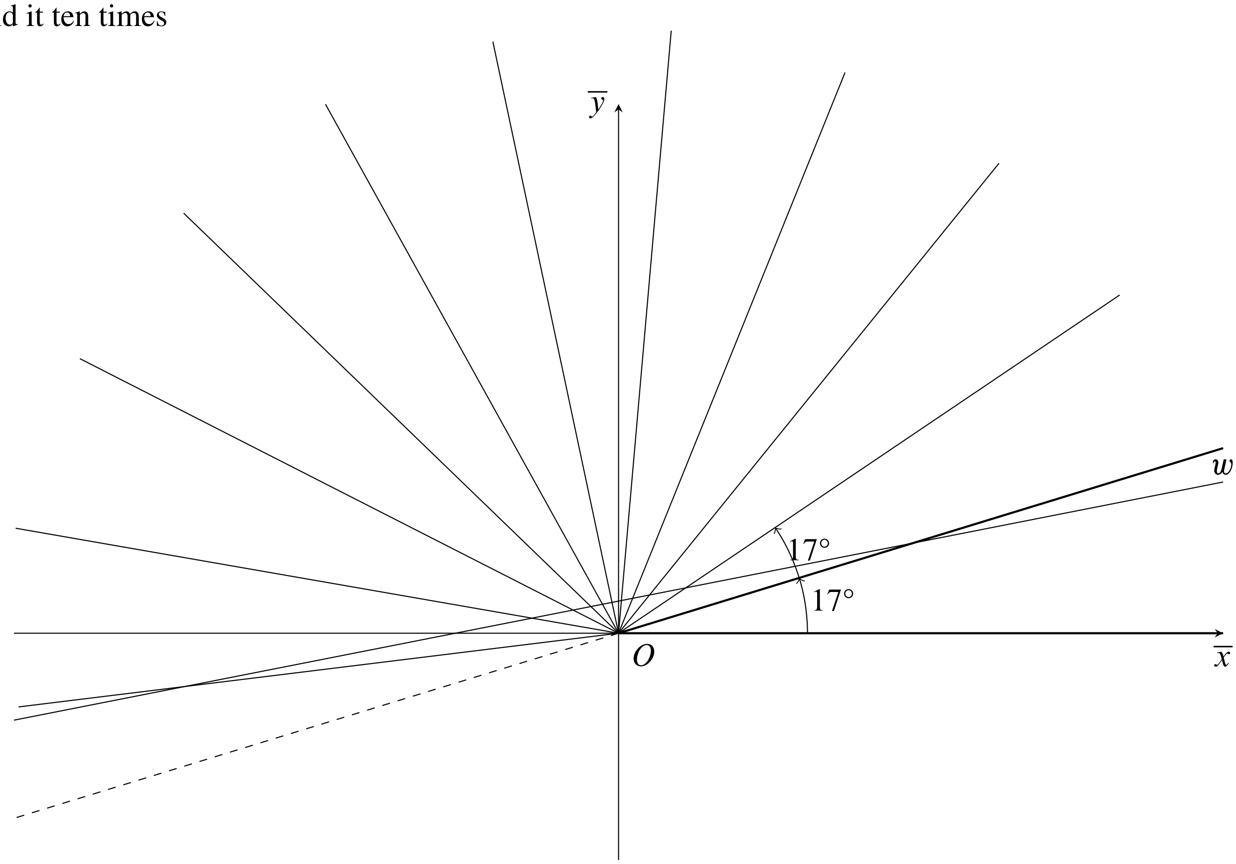
$$\begin{cases} \overrightarrow{\imath} \cdot (\overrightarrow{U} - \overrightarrow{V}) = 0 \\ ||\overrightarrow{U}||^2 = ||\overrightarrow{V}||^2 \end{cases}$$

where $\overrightarrow{\imath}$ is the unit vector on the $\bar{x}$-axis. Thus $\overrightarrow{U}$ and $\overrightarrow{V}$ are symmetric with respect to the $\bar{x}$-axis.

These two results prove that we can describe the mechanical system of the two masses on a half-line by the wedge $\bar{x}Ow$ with angle

$$\alpha = \arctan \frac{\sqrt{m_y}}{\sqrt{m_x}}$$

To draw a picture let us take $\alpha = 17°$, then $\tan\alpha = 0{,}30573068145866$ and $\frac{1}{\tan^2\alpha} = 10{,}698476851844558$. So let's suppose $m_y = 1$ and $m_x = 10{,}698476851844558$. Then we may draw the picture of the wedge and unfold it ten times



On this picture it is easy to see that the maximum number of shocks is 11 because $11 = \lceil \frac{180}{17} \rceil$. Here $\lceil \ \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function, such that $\lceil x \rceil$ is the smallest integer which is bigger or equal to x. If one denotes the integer part of x by $\lfloor x \rfloor$, then $\lceil x \rceil = -\lfloor -x \rfloor$.

The result is general : the maximum number of shocks is $\lceil \frac{\pi}{\alpha} \rceil$ if α is measured in radians. For instance if $m_x = 100^6 m_y$, $\lceil \frac{\pi}{\alpha} \rceil = 314159$, but it might be difficult to test it experimentally...