Chapitre 3

Billiard trajectories

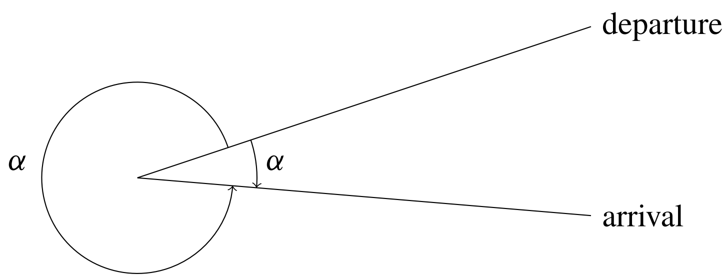
I.F.Fagnano posed in 1775 the following problem : "To inscribe in a given acute-angle triangle the triangle of a minimum perimeter".

§ 1. The Fagnano trajectory

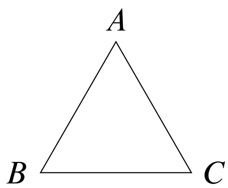
1.1. Preliminaries

Among the oriented angles in the plane we have

1. the angles between rays or vectors



In radians, the measure α of that angle belongs to $\mathbb{R}/2\pi\mathbb{Z}$, which means that α is an equivalence class modulo 2π. For instance in an equilateral triangle ABC (oriented positively)



the

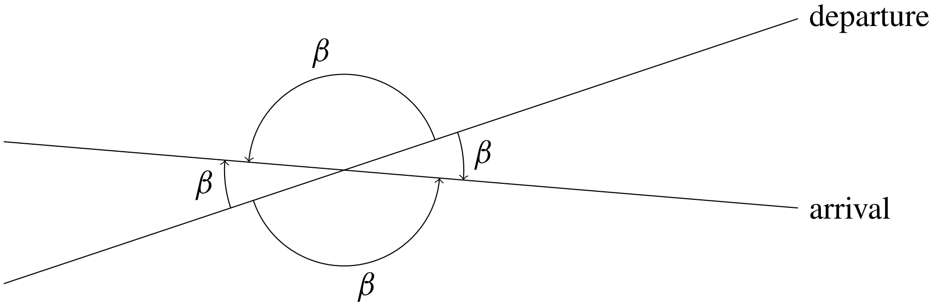
measure of the oriented angle of rays (or vectors) $(\overrightarrow{AB}, \overrightarrow{AC})$ is

$$\alpha = \left\{\dots, -\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \dots\right\}$$

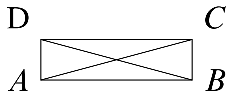
If you divide by 2 you get an element of $\mathbb{R}/\pi\mathbb{Z}$

$$\frac{1}{2}\alpha = \left\{\dots, -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \dots\right\}$$

2. the angles between lines



In radians, the measure β of that angle belongs to $\mathbb{R}/\pi\mathbb{Z}$, which means that α is an equivalence class modulo 2π. For instance in a rectangle $ABCD$ (oriented positively)



the measure of the oriented angle of lines between the two diagonals $(\overrightarrow{AB}, \overrightarrow{AC})$ is

$$\beta = \left\{\dots, -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \dots\right\}$$

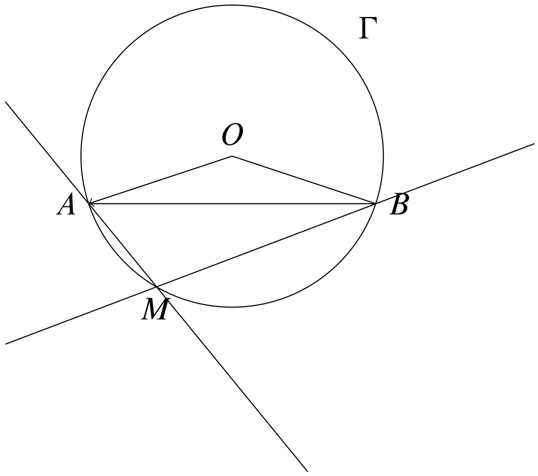
Think of two parallel lines, their angle is 0 modulo π or

$$\beta = \{k\pi; k \in \mathbb{Z}\}$$

.

Theorem 1. Let Γ be a circle with center O and let A and B be two distinct points belonging to Γ. A point M belongs to Γ if and only if

$$(MA, MB) = \frac{1}{2}(\overrightarrow{OA}, \overrightarrow{OB})$$



If you measure the angles on the picture you get $\alpha = \widehat{AOB} = 143.35°$ and $\beta = \widehat{AMB} = 108.32°$. How does it fit with $\beta = \frac{1}{2}\alpha$?

Définition. Points in an Euclidean plane are said to be *cocyclic* (or *concyclic*) if they lie to a common circle or a common line.

Theorem 2. Four points A, B, C and D are cocyclic if and only if the following equality between oriented angles of lines holds :

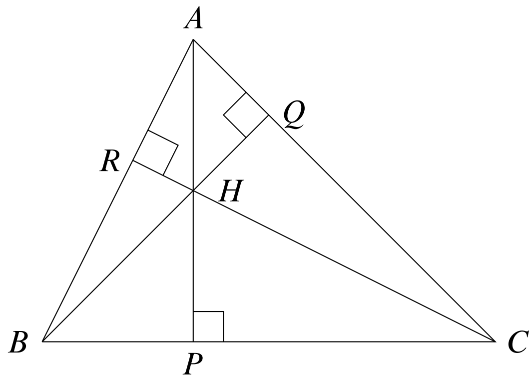
$$(CA, CB) = (DA, DB) \qquad (*)$$

Proof. Suppose the points are on a line, then $(CA, CB) = 0$ and $(DA, DB) = 0$. Thus $(*)$ holds. Suppose now the points are on a common circle. Let O be the center of that circle. Following the theorem 1 above, we have $(CA, CB) = \frac{1}{2}(\overrightarrow{OA}, \overrightarrow{OB})$ and similarly $(DA, DB) = \frac{1}{2}(\overrightarrow{OA}, \overrightarrow{OB})$, thus $(*)$ holds.

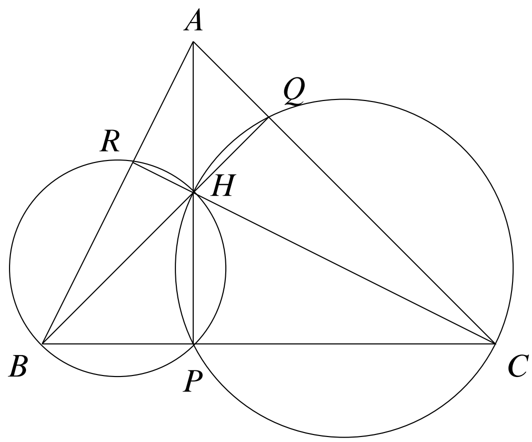
Conversely, suppose the four points verify $(*)$. Let Γ be the circle or line ABC. If Γ is a line, then $(CA, CB) = 0$, and from $(*)$ we have $(DA, DB) = 0$ which implies that D belongs to the line AB and the theorem is proved. If Γ is a circle, the converse of theorem 1 shows that D belongs to Γ. □

1.2. The orthic triangle of a triangle

Definition. Let ABC be a triangle. Let P be the point of BC such that AP is the height of that triangle issued from A. Similarly, let Q be the point of CA and R the point of AB such that BQ and CR are the two other heights. The triangle PQR is called the orthic triangle of the triangle ABC.

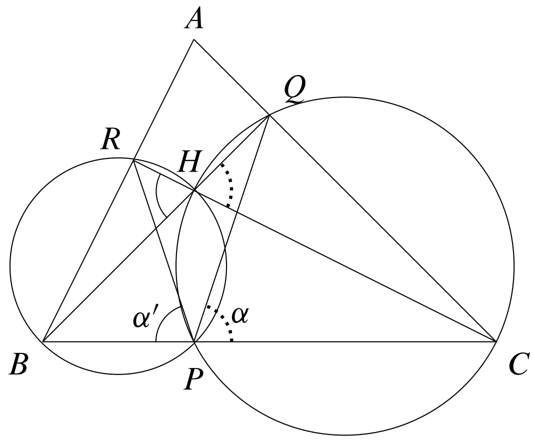


The three heights meet at a point H called *orthocenter*. Notice that since $(PH, PC) =$ a right angle $= (QH, QC)$, the points H, C, P and Q are on a common circle. Similarly, since $(RH, RB) = (PH, PB)$, there is a circle going through the four points H, B, R and P.



Theorem. Let PQR be the orthic triangle of a triangle ABC, then we have the following equalities of measures of angles

$$\widehat{CPQ} = \widehat{BPR} \qquad \widehat{AQR} = \widehat{CQP} \qquad \widehat{ARQ} = \widehat{BRP}$$



Proof. Let us call $\alpha = (PC, PQ)$ and $\alpha' = (PR, PB)$. Since the points C, Q, H and P are on common circle, we have $\alpha = (PC, PQ) = (HC, HQ)$. The line HC is the same as the line HR and the line HQ is the same as the line HB, thus $\alpha = (HR, HB)$. Since the four points H, R, B and P are cocyclic $(HR, HB) = (PR, PB)$ thus $\alpha = \alpha'$. □

1.3. Solving Fagnanos problem

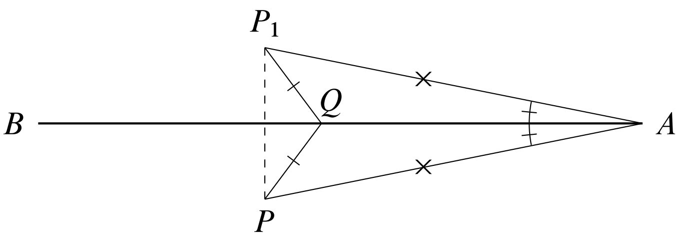
Definition. Let ABC be an acuteangled triangle. A triangle PQR is inscribed in the triangle ABC if P is a point on the segment BC, Q a point on CA and R a point on AB.

The perimeter p of a triangle PQR is the sum of the length of the sides of the triangle $p = |QR| + |RP| + |PQ|$.

Theorem. Let ABC be an acuteangled triangle. Among all the inscribed inscribed in ABC, the triangle with smallest perimeter is the orthic triangle.

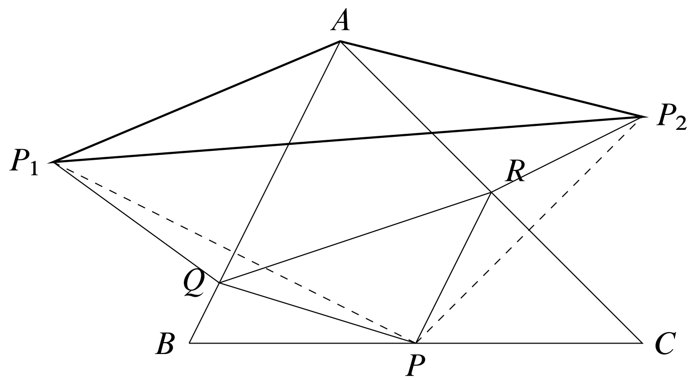
Proof. We do the proof in two steps. In the first step we suppose P fixed and we look for Q and R such that the perimeter p of the inscribed triangle PQR is minimum. In the second step we choose P along BC mnimising p.

First step. Let P_1 be the point image of P in the reflection along the line AB.



Notice that the symmetry gives us the equality of length $AP_1 = AP$ and the equality of angles $\widehat{BAP_1} = \widehat{PAB}$ and thus $\widehat{PAP_1} = 2\,\widehat{PAB}$. Notice also that since Q is a point on the axis of reflection, we have the equality of length $P_1Q = PQ$.

Let P_2 be the point image of P in the reflection along the line AC.



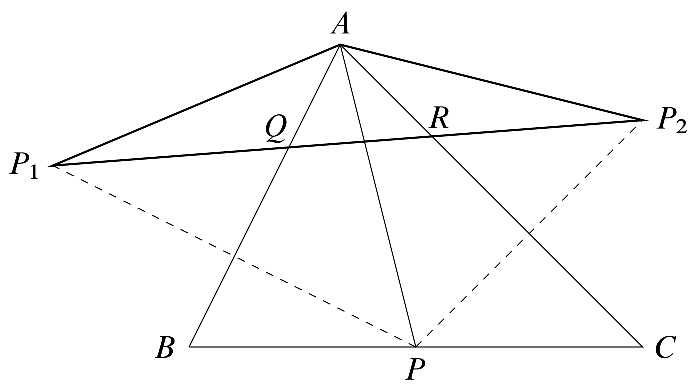
We get similarly $AP_2 = AP$, $\widehat{PAP_2} = 2\,\widehat{PAC}$ and $RP_2 = PR$.

Consider the triangle AP_1P_2. The angle $\widehat{P_1AP_2}$ is twice the angle $\widehat{BAC}$ at the vertex A of the initial triangle. That angle is constant. The triangle is isoceles since $AP_1 = AP = AP_2$.

The length of the broken line P_1QRP_2 is equal to the perimeter p of the inscribed triangle PQR since $P_1Q = PQ$ and $RP_2 = RP$ and thus

$$P_1Q + QR + RP_2 = PQ + QR + RP = p$$

Then for the fixed P the perimeter p will be minimal if the broken line P_1QRP_2 has minimal length. But we know that the shortest path from P_1 to P_2 is the straight line. Thus to find Q and R we just have to draw the line P_1P_2 and mark the intersections with the sides AB and AC.

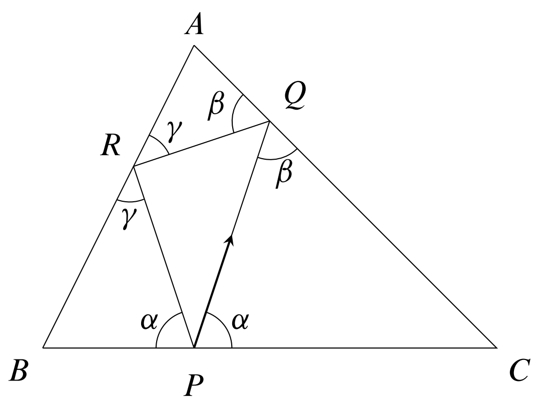


Second step. Now we have to choose the point P on the segment BC in such a way that $p = P_1 P_2$ is minimum. For each P we get a triangle $AP_1 P_2$. These triangles are all isosceles with same angle at the vertex A, namely $2\widehat{BAC}$. The basis $P_1 P_2$ will be minimum if the length of the equal sides is minimum. But the length of the sides is the length of the segment AP, and this segment has minimum length when P is the orthogonal projection of A on the side BC, that is when P is the foot of the height issued from the vertex A. But then by symmetry reasons Q and R have also to be the feet of the respective heights and thus PQR has to be the orthic triangle.QED □

1.4. The Fagnanos trajectory

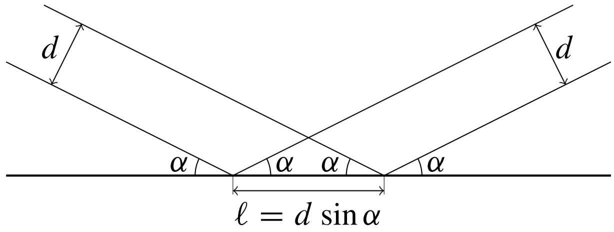
Let ABC be an acuteangled triangle and let us take it as the border of a billiard. Let PQR be the orthic triangle of ABC.

Let us start a trajectory at P in the direction of Q, when the ball arrives at Q it bounces following the reflection law and since the angles $\widehat{CQP}$ and $\widehat{RQA}$ are equal it goes in the direction of R and in R it bounces back into the direction of P coming back to the departure point after having run through 3 segments. After that the trajectory do again and again the same travel : The trajectory is said to be 3-*periodic*. Since it follows the border of the triangle solving the Fagnanos problem, it is called the Fagnano trajectory.



1.5. A family of 6-periodic trajectories

Let us first consider parallel trajectories bouncing on a common line



We see that the distance between the two trajectories before and after the shocks with the border are the same : it is an invariant in polygonal billiards.

Let us use the same billiard and start a trajectory from a point M on the segment PC strictly between P and C, not too far from P. We put $\ell = PM$. Let us start in the direction parallel to PQ. The billiard ball will bounce on the side AC in a point M_1 between Q and C such that MM_1 is parallel to PQ. Following the reflection law it will leave M_1 in a direction parallel to QR and meet the side AB in a point M_2 between R and B. Then it goes on to M_3 on BC between P and B such that M_2M_3 is parallel to RP.

The distance d between the parallel lines MM_1 and PQ is $d = \ell \sin\alpha$. This distance will be the same between M_1M_2 and QR and it will also be the distance between M_2M_3 and RP. Thus $PM_3 = \frac{d}{\sin\alpha} = \ell$.

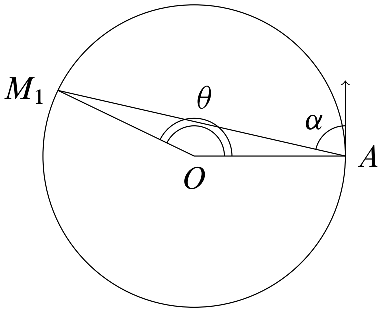
The trajectory continues after M_3 to $M4$ on CA, M_5 on AB and M_6 on BC. By the same reasoning as above we get $PM_6 = PM_3$, and thus $PM_6 = PM$. Since M and M_6 are on the same side of P on BC, we have $M_6 = M$. We also have $\widehat{PM_6M_5} = \widehat{BMM_1}$; thus the trajectory will start again towards M_1. Finally we have proved that the trajectory is 6-periodic.

§ 2. Circular billiards

Let D be limited by a circle. Let us start from a point A on the circle in a direction such that the angle of the first segment with the half-tangent in the positive direction is α measured in radians. We have two possibilities : the ratio $\frac{\alpha}{\pi}$ is or is not a rational number. We say that α is or is not π*-rational*.

2.1. α is π-rational

Let p and q be two integers which are relatively prime and such that $\alpha = \frac{p}{q}\pi$. The numbers p and q have no common divider other than 1 and thus the fraction $\frac{p}{q}$ cannot be simplified.



When the trajectory reaches the border for the first time in a point M_1, we have $\widehat{AOM_1} = 2\alpha$. Let us call this angle θ. Next point M_2 will be the image of M_1 in the rotation with center O and angle θ. After q bounces the image M_q will be such that the oriented angle of rays $(OA, OM_q) = q \times \frac{2p}{q}\pi = 2\pi p$. Thus M_q is the same point as A and the trajectory is q-periodic (and we also see that the star-shaped polygon has turned p times around the center.

2.2. α is π-irrational

The trajectory cannot be periodic since then it would be π-rational. But we have

Theorem. The vertices of a trajectory on a circle with angle α which is π-irrational is *equidistributed* on the circle.

Definition. Let I be any interval on the circle and denote by $|I|$ the length of that interval. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points on the circle and for each n call $k(n)$ the number of elements among the n first of the sequence belonging to I, that is

$$k(n) = \mathrm{Card}\{j \in \mathbb{N};\, j < n \quad \textbf{and} \quad x_j \in I\}$$

The sequence $(x_n)_{n\in\mathbb{N}}$ is equidistributed if

$$\lim_{n\to\infty} \frac{k(n)}{n} = \frac{|I|}{2\pi}$$

The theorem is a consequence of following theorem of Kronecker and Weyl. Let us call U the unit circle.

Theorem. If $f : U \to \mathbb{R}, x \mapsto f(x)$ is integrable function defined on the circle, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(x_j) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\mathrm{d}x$$

To see that the former theorem follows from this one just take f to be the characterstic function of the interval I

$$f(x) = \mathbb{1}_I(x) = \begin{cases} 1 & \text{if} \quad x \in I \\ 0 & \text{if} \quad x \notin I \end{cases}$$

and let the sequence be $x_n = x_0 + n\theta$.

Sketch of the proof of the theorem. One may approximate f by trigonometric polynomials that is linear combinations of functions $\cos kx$ and $\sin kx$ for $k \in \mathbb{Z}$. In fact it is easier to go to the complex functions since $\cos kx$ and $\sin kx$ are linear combinations of e^{ikx}. So the problem reduces to the proof for all k of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{ik(x_0+j\theta)} = \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} \mathrm{d}x \tag{*}$$

In the case where $k = 0$, the left side of the equality reduces to $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} 1 = \frac{1}{n} n = 1$ and the righthandside becomes $\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}x = 1$. In that case $(*)$ is verified.

In the case were $k \neq 0$, we have

$$\sum_{j=0}^{n-1} e^{ik(x_0+j\theta)} = e^{ikx_0} \sum_{j=0}^{n-1} \left\{ e^{ik\theta} \right\}^j$$

But one knows that $\sum_{j=0}^{n-1} a^j = \frac{1-a^n}{1-a}$ for any $a \neq 1$. Thus

$$\left| \sum_{j=0}^{n-1} e^{ik(x_0+j\theta)} \right| = \left| \frac{1 - e^{ikn\theta}}{1 - e^{ik\theta}} \right| \leqslant \left| \frac{2}{1 - e^{ik\theta}} \right|$$

and then the lefthand side of $(*)$ tends to 0 when $n \to \infty$. The righthandside of $(*)$ is also 0 since

$$\int_0^{2\pi} e^{ikx} \mathrm{d}x = \frac{1}{ik} \Big[e^{ikx} \Big]_0^{2\pi} = \frac{1}{ik} (1 - 1) = 0$$

§ 3. Elliptical billiards