Differential Projective Geometry and Schwarzian derivative

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The course follows the chapter called "ÉTUDE PROJECTIVE DE LA DROITE" in the book "Leçons sur LA THÉORIE DES ESPACES A CONNEXION PROJECTIVE" by Elie Cartan. In order to understand the main concepts we begin by recalling the general geometrical setting by a short description of the projecive geometries.

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Some Aspects of Projective Geometry

Chapitre 1

Projective spaces

§ 1. General definition of a projective space

1.1 Intrinsic definition

Defintion. Let V be a linear space (or vector space) over a field $\mathbb{K}$. The set of subspaces of dimension 1 of V is called the projective space derived from V and denoted PV. The elements of PV are called points.

Remark 1. If $V = \{0\}$, then $PV = \emptyset$. We suppose from now on that V contains vectors which are not equal to 0.

Remark 2. If $v \in V$ and $v \neq 0$, then the set $\mathbb{K}v = \{\lambda v \mid \lambda \in \mathbb{K}\}$ belongs to PV. Conversely, any point M in PV may be written as $\mathbb{K}v$ for some v in $V \setminus \{0\}$.

Proposition. Let f be a linear bijective map from a linear space V onto itself. The map f induces a map $\widetilde{f}$ defined by

$$\widetilde{f}(\mathbb{K}v) = \mathbb{K}f(v)$$

Proof. Let M be a point in PV. We choose any vector v in M different from 0. Then $\widetilde{f}(M)$ is the point $M' = \mathbb{K}f(v)$. This defines a map if and only if the point M' obtained is independent of the choice of v in $M \setminus \{0\}$. To prove that, let v_1 be any vector in $M \setminus \{0\}$; we have

$$v_1 = \mu v \quad \text{with } \mu \neq 0$$

and thus since f is linear $f(v_1) = \mu f(v)$. Then $\mathbb{K}v = \mathbb{K}v_1$ and $\mathbb{K}f(v) = \mathbb{K}f(v_1)$. $\square$

Definition. The maps $\widetilde{f}$ induced by linear bijective maps f are called *automorphisms of* PV, *projective linear transformation of* PV, *projective transformation of* PV or simply *homography*.

1.2 Relation between projective space and affine space

We consider a linear space V of finite dimension $n+1$. The projective space PV is said to be of dimension n.

Let φ be a linear map from V onto $\mathbb{K}$. The kernel W of φ is a linear subspace of V of dimension n :

$$W = \varphi^{-1}(0)$$

For each $\mu \in \mathbb{K} \smallsetminus \{0\}$, the subset W_μ of V defined by

$$W_\mu = \varphi^{-1}(\mu)$$

is called an *affine subspace of* V . Note that $W_\mu \cap W = \varnothing$ and $W_\mu \cap W_{\mu'} = \varnothing$ for $\mu \neq \mu'$. The affine subspaces W_μ are said *parallel* to each other and *parallel* to W.

Now look at a point M in PV ; either $M \subset W$ and $M \in PW$, or there is a vector v in M such that $\varphi(v) \neq 0$. In this second case there is a unique vector m in the affine subspace W_μ such that $M = \mathbb{K}m$. Let us identify the projective point M in $PV \smallsetminus W$ with the "point" m in the affine space W_μ such that $M = \mathbb{K}m$. Thus

$$PV = W_\mu \cup PW \qquad (*)$$

where the union is disjoint.

Conclusion : a projective space of dimension n is the disjoint union of an affine space of dimension n and a projective subspace PW of dimension $n-1$. The elements of PW are said to be *at infinity*.

Let us look at the formula $(*)$ for small values of n.

$n = 0$. The space V is a line.

The linear space V is of dimension 1. The subspace of dimension 0 is the set containing only the null vector also denoted 0, thus $W = \{0\}$ and $PW = \varnothing$. The only linear subspace of V of dimension 1, is V itself. Thus

$$PV = \{V\} \cup \varnothing$$

PV is a set containing only one point.

$n = 1$. The space V is a plane.

The linear space V is of dimension 2. For any subspace W of dimension 1 the projective subspace PW contains only one element, thus $W = \mathbb{K}m$ and $PW = \{M\}$. Thus

$$PV = \text{affine line} \cup \text{one point}$$

PV is a projective line.

$n = 2$. The space V is a 3-dimensional linear space.

The linear space V is of dimension 3. The projective space PV is of dimension 2 and called a projective plane. For any subspace W of dimension 1 the projective subspace PW is a projective line. Thus

$$\text{projective plane} = \text{affine plane} \cup \text{projective line}$$

1.3 Use of coordinates

Definition. We consider a linear space V of finite dimension $n+1$. Let $(e_1, \dots, e_{n+1})$ be a basis of V. Let M be a point belonging to PV. We call *homogeneous coordinates* of M any sequence of length $n+1$ of elements of $\mathbb{K}$ $(x_1, \dots, x_{n+1})$ such that

$$x_1e_1 + \cdots + x_{n+1}e_{n+1} \in V \setminus \{0\}$$

Proposition. Two sequences of length $n+1$ of elements of $\mathbb{K}$, $(x_1, \dots, x_{n+1})$ and $(y_1, \dots, y_{n+1})$ are homogeneous coordinates of a same point M in PV if and only if there is an element $\lambda \in \mathbb{K} \setminus \{0\}$ such that

$$\begin{cases} y_1 & = & \lambda x_1 \\ \dots & \dots & \dots \\ y_{n+1} & = & \lambda x_{n+1} \end{cases}$$

Remark. A point M in PV has $n+1$ homogeneous coordinates, but since they are defined up to a multiplicative constant, the point depends only on n parameters. Therefore it is natural to say that PV is of dimension n. If one tries to use just n numbers, one gets the coordinates of points in an affine plane. Thus some points (the points "at infinity") are forgotten. But sometimes it is nevertheless convenient to use such inhomogeneous coordinates.

Definition. Let W be the subspace of V with equation

$$x_{n+1} = 0$$

and W_1 the affine subset with equation

$$x_{n+1} = 1$$

For any point M in $PV \setminus PW$, we call *inhomogeneous coordinates* of M the sequence $(z_1, \dots, z_n)$ such that $(z_1, \dots, z_n, 1)$ are homogeneous coordinates of M.

Proposition. Let $(x_1, \dots, x_{n+1})$ be homogeneous coordinates of a point M in PV. We suppose that $M \notin W$, that is $x_{n+1} \neq 0$. Then the inhomogeneous coordinates of M are given by

$$\begin{cases} z_1 & = & \frac{x_1}{x_{n+1}} \\ \dots & \dots & \dots \\ z_n & = & \frac{x_n}{x_{n+1}} \end{cases}$$

Description of the homographies in coordinates

Let f be a bijective linear map of V onto itself. Given the basis $(e_1, \dots, e_{n+1})$, the map is described by a regular square matrix A of order $n+1$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,n+1} \end{bmatrix} \quad \text{with} \quad \det A \neq 0$$

The homography $\widetilde{f}$ is described by

$$\begin{cases} x'_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,n+1}x_{n+1} \\ \ldots & \ldots \quad \ldots \\ x'_n &= a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n+1}x_{n+1} \\ x'_{n+1} &= a_{n+1,1}x_1 + a_{n+1,2}x_2 + \ldots + a_{n+1,n+1}x_{n+1} \end{cases}$$

or in inhomogeneous coordinates

$$\begin{cases} z'_1 &= \frac{a_{11}z_1 + a_{12}z_2 + \ldots + a_{1,n+1}}{a_{n+1,1}z_1 + a_{n+1,2}z_2 + \ldots + a_{n+1,n+1}} \\ \ldots & \ldots \quad \ldots \\ z'_n &= \frac{a_{n,1}z_1 + a_{n,2}z_2 + \ldots + a_{n,n+1}}{a_{n+1,1}z_1 + a_{n+1,2}z_2 + \ldots + a_{n+1,n+1}} \end{cases}$$

Two matrices A and B describe the same homography if there is an element λ in $\mathbb{K} \setminus \{0\}$ such that

$$B = \lambda A$$

$n = 1$ Homographies of the projective line

When $n = 1$ the formulae above become

$$\begin{cases} x' &= ax + by \\ y' &= cx + dy \end{cases} \qquad \text{with} \quad ad - bc \neq 0$$

and

$$z' = \frac{az + b}{cz + d} \qquad \text{with} \quad ad - bc \neq 0$$

§ 2. Real affine line and real projective line

2.1 Directed distances and division ratio on a real affine line

A real affine line is the usual straight line without any unit or origin. Just a line :

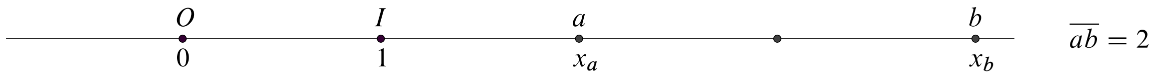
To describe points on such a line we need an origin, a distance and an orientation. In practice what we need is a frame which is a couple of distinct points (O, I). The abscissa x of a point m on the line is positive if m and I are on the same side of O, negative if m and I are on the two different rays with origin O. The absolute value is the quotient of the distance Om from O to m divided by the distance OI from O to I. We denote

$$x_m = \overline{Om}$$

where the frame (O, I) is not mentioned but is implicit.

Given two points a and b we define the directed distance $\overline{ab}$ by

$$\overline{ab} = x_b - x_a$$



If we change the frame (O, I), the directed distances will become different.

Change of frame on an affine line

Let (O', I') be another frame of the same real affine line. Let $x_{O'}$ and $x_{I'}$ be the abscissae of the points O' and I' in the old frame (O, I). The abscissa of a point a in this new frame can be written

$$x'_a = \frac{\overline{O'a}}{\overline{O'I'}} = \frac{x_a - x_{O'}}{x_{I'} - x_{O'}} = \alpha x_a + \beta \qquad \text{where } \alpha = \frac{1}{x_{I'} - x_{O'}} \text{ and } \beta = -\frac{x_{O'}}{x_{I'} - x_{O'}}$$

Definition and Proposition. Let a, b and c be three points on a line. We call *division ratio of the directed distances from c to a and b* the number

$$\frac{\overline{ca}}{\overline{cb}}$$

This number is independent of the choice of the frame.

Proof.

$$\left.\frac{\overline{ca}}{\overline{cb}}\right|_{\text{dans le repère } (O',I')} = \frac{x'_a - x'_c}{x'_b - x'_c} = \frac{(\alpha x_a + \beta) - (\alpha x_c + \beta)}{(\alpha x_b + \beta) - (\alpha x_c + \beta)} = \frac{x_a - x_c}{x_b - x_c} = \left.\frac{\overline{ca}}{\overline{cb}}\right|_{\text{dans le repère } (O,I)}$$

Example. The point c is the midpoint of the segment ab iff

$$\frac{\overline{ca}}{\overline{cb}} = -1$$

Remark. Let $x = \frac{\overline{ca}}{\overline{cb}}$, then

$$\frac{\overline{cb}}{\overline{ca}} = \frac{1}{x}\ ;\ \frac{\overline{ba}}{\overline{bc}} = 1 - x\ ;\ \frac{\overline{bc}}{\overline{ba}} = \frac{1}{1-x}\ ;\ \frac{\overline{ac}}{\overline{ab}} = \frac{x}{x-1}\ ;\ \frac{\overline{ab}}{\overline{ac}} = 1 - \frac{1}{x}$$

2.2 Homographies of the real projective line

Recall that the real projective line ℓ may be described by a real affine line to which is added one point called *point at infinity*. The abscissa of the point at infinity is denoted ∞. Thus ℓ is in bijection with $\mathbb{R} \cup \{\infty\}$.

Definition

The automorphism of ℓ are the homographies described by the bijective maps $f : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}$ such that $ad - bc \neq 0$ and

$$\begin{cases} \text{if} \quad z \in \mathbb{R} \quad \text{then} \quad f(z) = \frac{az+b}{cz+d} \\ \text{if} \quad z = \infty \quad \text{then} \quad f(z) = \frac{a}{c} \text{ if } c \neq 0 \text{ and } \text{ and } f(z) = \infty \text{ if } c = 0 \end{cases}$$

Classification of real homographies

A *fixed point* of a homography f is an element z of $\mathbb{R} \cup \{\infty\}$ such that $z = f(z)$. Let $f(z) = \frac{az+b}{cz+d}$, with $c \neq 0$, then z is a fixed point iff

$$cz^2 + (d-a)z - b = 0$$

It is an equation of degree 2 with $\Delta = (d-a)^2 + 4bc$: the number of solutions is 2 when $\Delta > 0$, 1 when $\Delta = 0$ and 0 when $\Delta < 0$.

Notice that if $c = 0$, then $f(\infty) = \infty$; thus ∞ is a fixed point. The equation for fixed point becomes

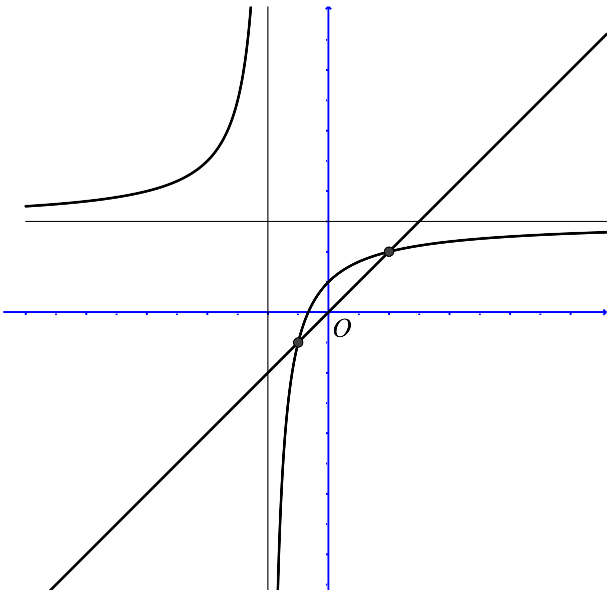
$$(d-a)z - b = 0$$

Either $d \neq a$ and we have one fixed point other than ∞, or $d = a$ and then f is a translation

$$f(z) = z + \frac{b}{d}$$

This equation has no solution in $\mathbb{R}$, but we can think of ∞ as a solution. The homography f has then 1 fixed point and we may view ∞ as a double solution.

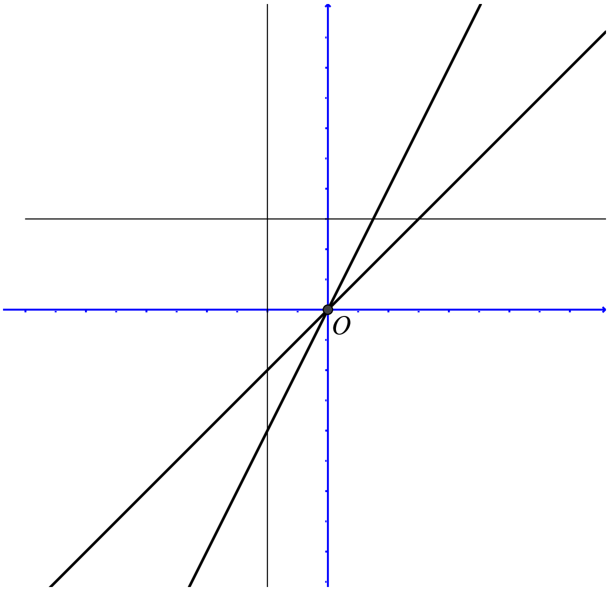
Definition. A homography of $\mathbb{R} \cup \{\infty\}$ is called *hyperbolic* if it has 2 fixed points, *parabolic* if it has 1 fixed point and *elliptic* if it has 0 fixed point.



Hyperbolic homography

$z \longmapsto \frac{3z+2}{z+2}$

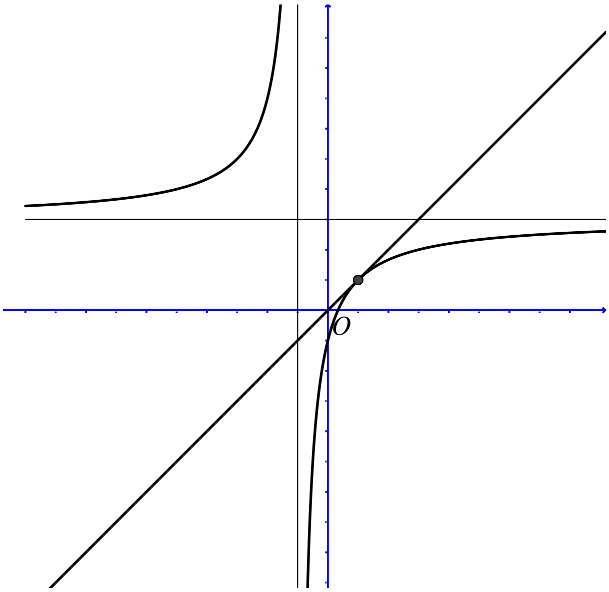
Fixed points : -1 and 2



Hyperbolic homography

$$z \longmapsto 2z$$

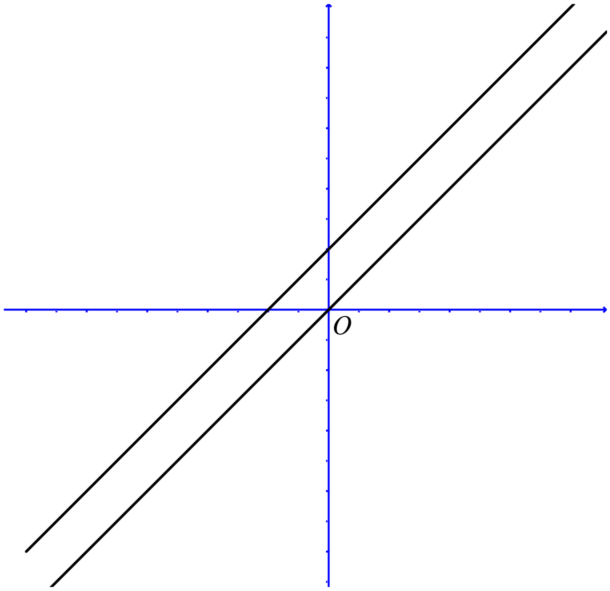
Fixed points : 0 and ∞



Parabolic homography

$$z \longmapsto \frac{3z-1}{z+1}$$

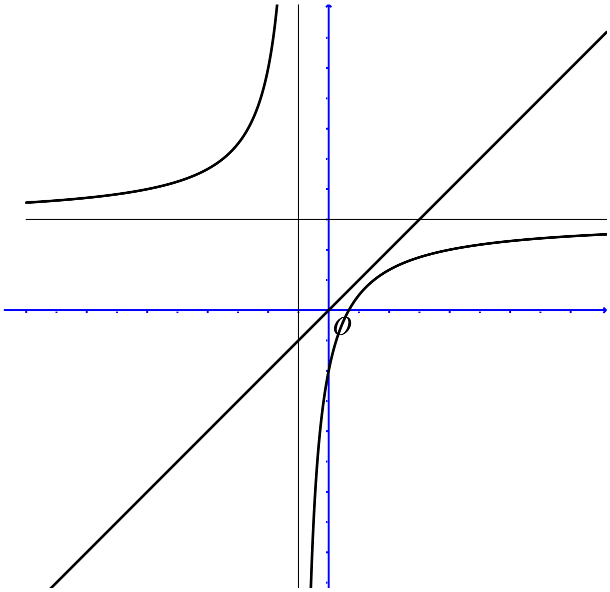
Fixed point : 1 (double)



Parabolic homography

$$z \longmapsto z + 2$$

Fixed point : ∞ (double)



Elliptic homography

$$z \longmapsto \frac{3z-2}{z+1}$$

Fixed point : none

2.3 cross-ratios

Cross-ratio of 4 elements of $\mathbb{R} \cup \{\infty\}$

Definition. Let z_1, z_2, z_3 and z_4 be four elements of $\mathbb{R} \cup \{\infty\}$, the cross-ratio of these four (generalized) numbers denoted by $(z_1, z_2; z_3, z_4)$ is given by

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1} \frac{z_4 - z_2}{z_3 - z_2}$$

Remark 1. We can write the cross-ratio as a ratio of ratios like

$$(z_1, z_2; z_3, z_4) = \frac{\frac{z_3 - z_1}{z_4 - z_1}}{\frac{z_3 - z_2}{z_4 - z_2}}$$

Therefore the cross-ratio is also called "birapport" in french. Other names for the cross-ratio are "double ratio" and "anharmonic ratio".

Remark 2. If one of the elements is ∞ you simplify with the two factors containing ∞ :

$$(\infty, z_2; z_3, z_4) = \frac{z_4 - z_2}{z_3 - z_2} \;\; ; \;\; (z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1} \;\; ; \;\; (z_1, z_2; \infty, z_4) = \frac{z_4 - z_2}{z_4 - z_1} \;\; \text{and} \;\; (z_1, z_2; z_3, \infty) = \frac{z_3 - z_1}{z_3 - z_2}$$

Cross-ratio of 4 points on a line

Definition. Let ℓ be a projective line. Choose one point m_∞ in ℓ and call it point at infinity and let Δ be the affine line $\ell \smallsetminus \{m_\infty\}$. Choose a frame (O, I) on Δ. We call cross-ratio of four points a, b, c and d on Δ the cross-ratio of the abscissae :

$$(a, b; c, d) := (z_a, z_b; z_c, z_d) = \frac{z_c - z_a}{z_d - z_a} \frac{z_d - z_b}{z_c - z_b}$$

We extend the definition to the cases when one of the four points a, b, c or d is the point m_∞ by giving to the point m_∞ the abscissa ∞.

Theorem. A bijection f of $\mathbb{R} \cup \{\infty\}$ preserves the cross-ratios if and only if f is a homography.

Proof. Let f be such that for all z

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \qquad \text{where} \quad \alpha\delta - \beta\gamma \neq 0$$

and let us compute

$$\begin{aligned}
(f(z_a), f(z_b); f(z_c), f(z_d)) &= \frac{\frac{\alpha z_c+\beta}{\gamma z_c+\delta} - \frac{\alpha z_a+\beta}{\gamma z_a+\delta}}{\frac{\alpha z_d+\beta}{\gamma z_d+\delta} - \frac{\alpha z_a+\beta}{\gamma z_a+\delta}} \frac{\frac{\alpha z_d+\beta}{\gamma z_d+\delta} - \frac{\alpha z_b+\beta}{\gamma z_b+\delta}}{\frac{\alpha z_c+\beta}{\gamma z_c+\delta} - \frac{\alpha z_b+\beta}{\gamma z_b+\delta}} \\
&= \frac{(\alpha z_c+\beta)(\gamma z_a+\delta)-(\alpha z_a+\beta)(\gamma z_c+\delta)}{(\alpha z_d+\beta)(\gamma z_a+\delta)-(\alpha z_a+\beta)(\gamma z_d+\delta)} \frac{(\alpha z_d+\beta)(\gamma z_b+\delta)-(\alpha z_b+\beta)(\gamma z_d+\delta)}{(\alpha z_c+\beta)(\gamma z_b+\delta)-(\alpha z_b+\beta)(\gamma z_c+\delta)} \\
&= \frac{(\alpha\delta-\beta\gamma)(z_c-z_a)}{(\alpha z_d+\beta)(\gamma z_a+\delta)-(\alpha z_a+\beta)(\gamma z_d+\delta)} \frac{(\alpha z_d+\beta)(\gamma z_b+\delta)-(\alpha z_b+\beta)(\gamma z_d+\delta)}{(\alpha z_c+\beta)(\gamma z_b+\delta)-(\alpha z_b+\beta)(\gamma z_c+\delta)} \\
&= \frac{z_c-z_a}{z_d-z_a} \frac{z_d-z_b}{z_c-z_b} \\
&= (z_a, z_b; z_c, z_d)
\end{aligned}$$

Conversely, let f be a bijection which preserves cross-ratios. Once we have the three distinct images $f(z_a)$, $f(z_b)$ and $f(z_c)$ of three distinct numbers z_a, z_b and z_c, we have for any z

$$(f(z), f(z_a); f(z_b), f(z_c)) = (z, z_a; z_b, z_c)$$

expressed also as

$$\frac{f(z_b) - f(z)}{f(z_d) - f(z)}\Phi = \frac{z_b - z}{z_c - z}\varphi$$

where $\Phi = \frac{f(z_d)-f(z_a)}{f(z_b)-f(z_a)}$ and $\varphi = \frac{z_d-z_a}{z_b-z_a}$ are constants. Thus :

$$(z_c - z)(f(z_b) - f(z))\Phi = (z_b - z)\varphi(f(z_d) - f(z))$$

that is :

$$[\varphi(z_b - z) - \Phi(z_c - z)]f(z) = \varphi f(z_d)(z_b - z) - \Phi f(z_b)(z_c - z)$$

and finally :

$$f(z) = \frac{[\Phi f(z_b) - \varphi f(z_d)]z + [\varphi z_b f(z_d) - \Phi z_c f(z_b)]}{[\Phi - \varphi]z + [\varphi z_b - \Phi z_c]}$$

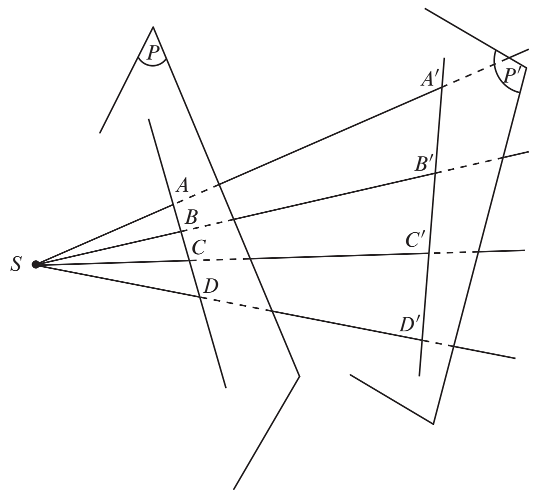
Thus f is a homography. □

2.4 Why is projective geometry called projective ?

Let E be the usual 3-dimensional space, let P be any plane and S any point which doesn't belong to P. The central projection of E on P with center S is the map that associates to any point M in S the intersection m of the line CM and the plane P. This definition is not so good since there are points in E which do not have any image through this projection. To avoid this difficulty the notion of points and lines *at infinity* were introduced...

Now we can restrict ourselves and consider projection with center S from one plane P on an other plane P' (of course S should not belong to P nor to P'). It is clear that the image of a line will be a line and also that intersections will become intersections.

What else is preserved ?



Answer : cross-ratios. But to look at that property it is enough to look at central projections in a plane of one line onto an other line.

2.5 Real projective plane

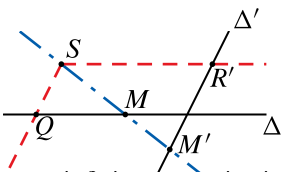
The real projective plane geometry is the study of figures and properties preserved by central (inclusive parallel) projection of a plane on an other. Such projections preserve lines and cross-ratios of aligned points. But to get bijections one adds to each plane P not only one point at infinity but a *line at infinity*. The points of this line are the directions of the lines in P (the direction of a line may be defined as the equivalence class relative to parallelism ; thus the direction of a line ℓ is the set of all the lines parallel to ℓ). Let us denote ∞_P the line at infinity of the plane P. Thus two parallel and distinct lines ℓ and ℓ' intersect in one point at infinity since they have the same direction. We add this point at infinity to the line ℓ ; but notice : there is only one line at infinity on a line ℓ. It is the same point "at both ends" : the line is like a circle !

If d is a line we get the associated projective line Δ by adding a point at infinity which we denote ∞_Δ. Thus

$$\Delta = d \cup \{\infty_\Delta\} \quad \text{and} \quad \{\infty_\Delta\} = \Delta \cap \infty_P$$

Definition. Let Δ and Δ' be two projective lines and S a point in the projective plane P. We suppose that S does not belong to Δ nor to Δ'. Let Q be the intersection of the parallel to Δ' with Δ, let R' be the intersection of the parallel to Δ with Δ' (if Δ and Δ' are parallel then $Q = R' = \infty_\Delta = \infty\Delta'$). The *central projection* with center S from Δ on Δ' is defined as follows :

$$\Delta \longrightarrow \Delta', M \longmapsto M' \text{ such that } \begin{cases} \text{if } M \neq \infty_\Delta \text{ and } M \neq Q & \text{then } M' = \Delta' \cap SM \\ \text{if } M = \infty_\Delta & \text{then } M' = R' \\ \text{if } M = Q & \text{then } M' = \infty_{\Delta'}. \end{cases}$$



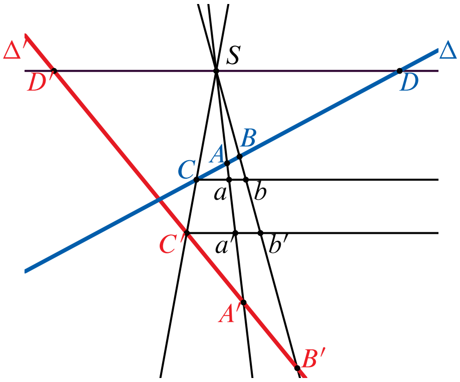
By the introduction of the points at infinity our projection is a bijection.

Theorem. Central projections preserve cross-ratios.

Proof. Let S be a point, Δ and Δ' two lines which do not contain S. Let A, B, C and D be four points belonging to Δ. The line Δ' intersects SA in A', SB in B', SC in C' and SD in D'. We want to show

$$(A, B; C, D) = (A', B'; C', D') \tag{1}$$

Let us draw the parallels to the line SD going through C and C'. These lines intersect SA in a and a' and SB in b and b'.



Since the triangles ACa and ADS are similar, we have $\dfrac{\overline{AC}}{\overline{AD}} = \dfrac{\overline{aC}}{\overline{SD}}$. Similarly since the triangles BCb and BDS are similar, we have $\dfrac{\overline{BD}}{\overline{BC}} = \dfrac{\overline{SD}}{\overline{bC}}$. From that we get

$$(A, B; C, D) = \frac{\overline{AC}}{\overline{AD}}\,\frac{\overline{BD}}{\overline{BC}} = \frac{\overline{aC}}{\overline{SD}}\,\frac{\overline{SD}}{\overline{bC}} = \frac{\overline{aC}}{\overline{bC}} \tag{2}$$

In the same way we show

$$(A', B'; C', D') = \frac{\overline{a'C'}}{\overline{b'C'}} \tag{3}$$

Consider the homothety (or homothecy or homogeneous dilation or homothetic transformation) with center S which transforms C into C'. It transforms a into a' and b into b'. Thus :

$$\frac{\overline{aC}}{\overline{bC}} = \frac{\overline{a'C'}}{\overline{b'C'}} \tag{4}$$

From (2), (3) and (4) we get (1). □

As a consequence of this theorem, we see that if we cut a pencil of four concurrent lines[1] by a line Δ the cross-ratio of the four points on Δ is independent of the choice of Δ. This cross-ratio can then be thought of as belonging to the pencil.

Definition. Let Δ_a, Δ_b, Δ_c and Δ_d be four lines going through a common point S. The cross-ratio of these four lines in that order denoted $(\Delta_a, \Delta_b; \Delta_c, \Delta_d)$ is equal to the number $(A, B; C, D)$ where A, B, C and D are the intersection points of any line Δ with the four lines Δ_a, Δ_b, Δ_c and Δ_d.

Remark 1. This shows why it is possible to define the cross-ratio of four points on a projective line : these four points are in fact four coplanar lines through the origin O of 2-dimensional linear space.

Remark 2. To define the homographies on a line Δ, one may proceed in the following way : make a central projection f_1 from Δ onto an other line Δ_1 and then a central projection f_2 from Δ_1 onto a line Δ_2 and so on. After n such projections make a central projection

[1] Lines are *concurrent* if they have a common point ; a *pencil* of lines is a set of concurrent lines.

f_{n+1} from Δ_n onto Δ. The map $f = f_{n+1} \circ \ldots f_2 \circ f_1 : \Delta \longrightarrow \Delta$ can be defined as a "homographic bijection" of the projective line Δ on itself. Since the cross-ratios are preserved at each central projection, f preserves cross-ratios and because of the theorem above f is a map that can be described by

$$\Delta \longrightarrow \Delta,\ z \longmapsto f(z) = \frac{az+b}{cz+d}$$

There is still a question : do we get all possible homographies in this way ? We can simplify the question if we recall that a homography is characterized by the images of any three distinct points. Thus the question may be put as follow : let A, B and C be three distinct points of a projective line Δ in a projective plane P and let A', B' and C' be three distinct points of the projective line Δ. Can we find central projections $f_1 : \Delta \longrightarrow \Delta_1$, $f_2 : \Delta_1 \longrightarrow \Delta_2$ and $f_3 : \Delta_2 \longrightarrow \Delta$ such that the images of A, B and C by $f_2 \circ f_1$ are respectively A', B' and C' ? The answer is yes, but we leave it as an exercise !

2.6 Harmonic division

The *harmonic division* is the generalization to projective geometry of midpoint in affine geometry.

Definition. Four points A, B, A' and B' in that order constitute a *harmonic division* if their cross-ratio is equal to -1.

A, B, A' and B' constitute a harmonic division $\Longleftrightarrow (A, B, A', B') = -1 \Longleftrightarrow \dfrac{\overline{A'B}}{\overline{A'A}} = -\dfrac{\overline{B'B}}{\overline{B'A}}$

Equivalent formulations.

- $(A, B; A', B') = -1$
- $(A, B; B', A') = -1$
- $(A', B'; A, B) = -1$

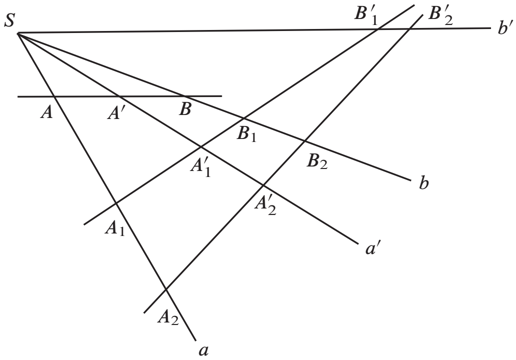
...

- $(a+b)(a'+b') = 2(ab+a'b')$
- $IA^2 = IB^2 = \overline{IA'}\,\overline{IB'}$, where I is the midpoint of the segment AB
- $\overline{AB}$ est la moyenne harmonique de $\overline{AA'}$ et de $\overline{AB'}$, soit

$$\frac{2}{\overline{AB}} = \frac{1}{\overline{AA'}} + \frac{1}{\overline{AB'}}$$

Harmonic pencil of four lines. Four conccurrent lines form a *harmonic pencil* if the cross-ratio of these four lines is -1.

Proposition. A pencil of four concurrent lines a, b, a' and b' is harmonic if and only if a line d parallel to b' intersects a, b and a' in points A, B and A' such that A' is the middle of the segment AB.



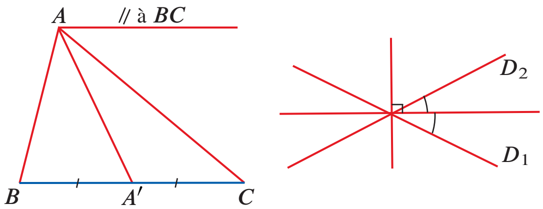
Proof. When the point B' is at infinity the cross-ratio becomes a ratio of directed distances

$$(A, B; A', \infty) = \frac{\overline{AA'}}{\overline{A\infty}} \frac{\overline{B\infty}}{\overline{BA'}} = \frac{\overline{AA'}}{\overline{BA'}}$$

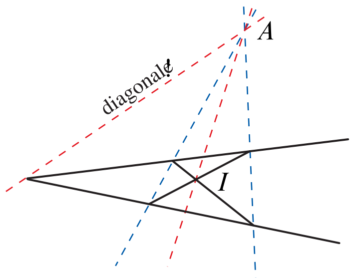
and this ratio is equal to -1 if and only if A' is the midpoint of the segment AB. □

Examples of harmonic pencils.

- Let A be a vertex of a triangle ABC. Two sides AB and AC, the median AA' issued from A and the parallel through A to the side BC is a harmonic pencil.
- The bisectors and the sides of an angle constitute a harmonic pencil.

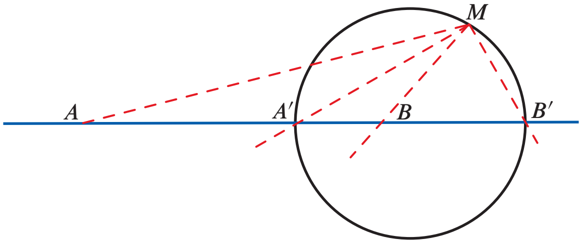


- Let A be a vertex in a complete quadrilateral (that is four lines such that no three lines are concurrent). Let I be the intersection of the two other diagonals. The pencil of four lines formed by the two sides through A, the diagonal through A and the line AI is harmonic.



- *Apollonius' circle.* Let A and B be two distinct points and let k be a positive number. The set of points M such that $\frac{MA}{MB} = k$ is the circle with diameter $A'B'$, where A' and B' are the points of the line AB which divide the segment AB in the ratio k.

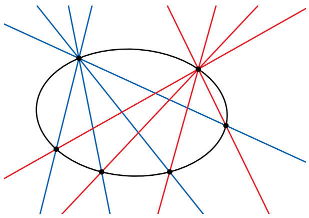
Since the four points A, B, A' and B' form harmonic division, the pencil $(MA, MB; MA'MB')$ is harmonic.



Remark 1. What happens when $k = 1$? The point A' becomes the middle of the segment AB, B' the point at infinity, MA' the bisector of the segment AB and MB' is the parallel to AB through M. The « circle » is then the line bisecting the segment AB.

Exercise. Let M, N, A, B, C et D be six points on a conic. Show that

$$(MA, MB; MC, MD) = (NA, NB; NC, ND)$$



§ 3. Complex affine line and complex projective line

3.1 Division ratio on a complex affine line

A complex affine line is in bijection with $\mathbb{C}$. There are plenty of such bijections. A bijection is fixed as soon as we have the images of two distinct points. Usually the most common choice is to choose a point associated to the number 0 and a point I associated to the number 1. The couple (O, I) is called a *frame* of the complex line. We denote by z_M the complex number associated with the point M with respect to a given frame. Notice that the complex line looks like a plane since it is in bijection with $\mathbb{R}^2$!

If we have three points A, B and C, we may compute the *division ratio*

$$z_{\text{triangle } ABC} = \frac{z_C - z_A}{z_B - z_A}$$

Let T be a point such that $z_T = z_{\text{triangle } ABC}$. Then the triangles ABC and OIT are similar. Thus every complex number characterise a class of similar triangles. Notice that the three points are aligned if $z_{\text{triangle } ABC} \in \mathbb{R}$.

If we have three distinct points A, B and C, they are the vertices of 6 distinct triangles. If z is the division ratio associated to one, then the division ratios associated to the 5 others are

$$\frac{1}{z}\ ;\ 1-z\ ;\ \frac{1}{1-z}\ ;\ \frac{z}{z-1}\ ;\ 1-\frac{1}{z}$$

We recognize once again the group of permutations of three objects $\mathcal{S}_{\ni}$.

3.2 Homographies of a complex projective line

The complex projective line may be obtained from the affine line by adding ONE POINT AT INFINITY denoted ∞ (remember that we add a complete projective line to the real affine plane to get the real projective plane). Thus the complex projective line has the shape of the Riemann sphere. Let us suppose we have a frame (O, I) on the affine complex line ℓ. We denote the projective complex line by Δ. Thus $\Delta = \ell \cup \{\infty\}$. A homography of the complex projective line Δ is a bijection f of Δ onto itself such that there are four complex numbers a, b, c and d such that $ad - bc \neq 0$ and

$$\begin{cases} \text{if} \quad z \in \mathbb{R} \quad \text{then} \quad f(z) = \frac{az+b}{cz+d} \\ \text{if} \quad z = \infty \quad \text{then} \quad f(z) = \frac{a}{c} \text{ if } c \neq 0 \text{ and } \text{ and } f(z) = \infty \text{ if } c = 0 \end{cases}$$

If we use homogeneous coordinates x and y to describe the points on the line, we have

$$x \in \mathbb{C} \quad y \in \mathbb{C} \quad \text{where } (x, y) \neq (0, 0) \text{ and } z = \frac{x}{y}$$

with the convention that if $y = 0$ (and then $x \neq 0$), then $z = \infty$. The homographies are then described by

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{where } (a, b, c, d) \in \mathbb{C}^4 \text{ and } ad - bc \neq 0$$

Remark 1. A homography is depending on 3 complex parameters (or 6 real parameters). To characterize a homography you need then just to know the images of three distinct points.

Remark 2. A homography of the complex projective line is also called a Möbius transformation.

Classification of the homographies of the complex projective line

The classification corresponding to the classification of real homographies is easier than in the real case. Here we have only two possibilities : either the second order equation has a double solution and the homography is called *parabolic* or it has two distinct solutions. The homographies with two fixed points are not all called hyperbolic.

The parabolic homography has double fixed point $z_0 \neq \infty$ if $cz_0^2 + (d-a)z_0 - b = 0$ and $2cz_0 + d - a = 0$. Thus $z_0 = \frac{a-d}{2c}$. Now if $c = 0$ the solution has to be ∞ and the equation $(d-a)z - b = 0$ has to have the solution ∞. Thus $d - a = 0$ and the homography has the form

$$z' = \frac{az+b}{a}$$

that is $z' = z + h$ which is a translation. By changing the frame in a complex projective line we can always write a parabolic homography as a translation.

If we have a homography with two fixed points, let us take a frame such that the fixed points are 0 and ∞. For that we need respectively $b = 0$ and $c = 0$.Then the homography takes the form $z' = \frac{az}{d}$ with $ad \neq 0$ or

$$z' = kz \quad \text{with } k \neq 0$$

It is just a bijective homothety (or homothecy, or homogeneous dilation). We have to exclude $k = 1$ which describes the identity map. If k is real positive but different from 1, the homography is called *hyperbolic*. If λ is such that $|k| = 1$, but $k \neq 1$ then the homography is called *elliptic*. If $k \in \mathbb{C} \setminus (\mathbb{R}^+ \cup \{z; |z| = 1\})$ then the homography is called *loxodromic*.

To the homography $z' = \frac{az+b}{cz+d}$ we associate the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. By a change of frame on the projective line the matrix is changed into a conjugate matrix and all the conjugate matricies may be obtained. A regular complex square matrix of order 2 is conjugated either to the matrix $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ or to a matrix $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda \neq 0$. But two matrices A and A' such that $A = \alpha A'$ with $\alpha \neq 0$ are associated to the same homography. We can thus describe the classification in the following way : the homography different from the identity is associated to

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and is then called *parabolic*
2. $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $|\lambda| = 1$ and $\lambda \neq 1$, and is then called *elliptic*
3. $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $|\lambda| \in \mathbb{R} \setminus \{-1, 0, 1\}$, and is then called *hyperbolic*
4. $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $|\lambda| = 1$ and $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \{z \in \mathbb{C} \mid |z| = 1\})$, and is then called *loxodromic* (etymology : (dromos) running (loxo) slantwise).

3.3 cross-ratios

Cross-ratio of 4 elements of $\mathbb{C} \cup \{\infty\}$

Definition. Let z_1, z_2, z_3 and z_4 be four elements of $\mathbb{C} \cup \{\infty\}$, the cross-ratio of these four (generalized) complex numbers denoted by $(z_1, z_2; z_3, z_4)$ is given by

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1} \frac{z_4 - z_2}{z_3 - z_2}$$

If one of these elements is ∞ you have to simplify away the two factors where it appears. For example :

$$(z_1, \infty; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1}$$

or

$$(z_1, z_2; \infty, z_4) = \frac{z_4 - z_2}{z_4 - z_1}$$

Cross-ratio of 4 points on a complex projective line

The definition is the same as in the real case.

Definition. Let ℓ be a complex projective line. Choose one point m_∞ on ℓ and call it point at infinity. Let Δ be the affine line $\ell \smallsetminus \{m_\infty\}$. Choose a frame (O, I) on Δ. We call cross-ratio of the four points a, b, c and d on Δ the cross-ratio of the abscissae :

$$(a, b; c, d) := (z_a, z_b; z_c, z_d) = \frac{z_c - z_a}{z_d - z_a}\frac{z_d - z_b}{z_c - z_b}$$

We extend the definition to the cases when one of the four points a, b, c or d is the point m_∞ by giving to the point m_∞ the abscissa ∞.

Theorem. A bijection f of $\mathbb{R} \cup \{\infty\}$ preserves the cross-ratios if and only if f is a homography.

Proof. The same as in the real case. □

Theorem. The cross-ratio of four complex numbers is real if and only if the four points belong to a common circle or real line.

Proof. Let A, B, C and D be four points on the complex affine line and let the complex numbers associated to these points with respect to an affine frame (O, I) be z_A, z_B, z_C and z_D. Now we notice that

$$\arg\left(\frac{z_C - z_A}{z_D - z_A}\right) = \text{oriented angle}(\overrightarrow{AD}, \overrightarrow{AC})$$

Then the cross-ratio is real if and only if

$$(\overrightarrow{AD}, \overrightarrow{AC}) - (\overrightarrow{BD}, \overrightarrow{BC}) = 0 \quad (\text{mod } \pi)$$

which characterize the fact that the four points belong to a circle or a real line. End the proof by looking at what happens when one of the points is ∞. □

Chapitre 2

Reading Elie Cartan :
La droite projective réelle

§ 1. Projective equality of linear motions. Schwarzian

We want to study curves from a cinematic point of vue, that is to say parametric curves. Two such curves are "equal" if not only the geometric curves are equal but also if the mobile point is located at similar points for equal value of the parameter t.

In Euclidean geometry let z be the real abscissa of a point on a real oriented Euclidean affine line with respect to a frame (O, I) where the distance OI is equal to 1 and the direction from O to I is direct. Let us consider two parametric curves $z = f(t)$ and $z = F(t)$. When are they "equal" ? We should have $F(t) = f(t) + b$ for some b. The condition may be written

$$\forall t \in \mathbb{R} \quad F'(t) = f'(t)$$

The two mobile points have same speed. The problem was easily solved since the group of invariance of the oriented Euclidean line is just the group of translations. In the case of the projective line the group is the group of homographies.

We'l denote by Δ the projective line on which the mobile point is moving. Let (A, B, C) be three distinct points of Δ. We take this triplet (A, B, C) as a frame on Δ, which means that the abscissa z of the mobile point M is the following cross-ratio :

$$z = (M, A; B, C)$$

In such a frame we have $z_A = 1$, $z_B = 0$ and $z_C = \infty$. Thus :

$$(z, 1; 0, \infty) = (M, A; B, C)$$

Two motions (or parametric curves)

$$z = f(t) \quad \text{and} \quad z = F(t)$$

are "equal" if and only if there is a homography $z \longmapsto \frac{az+b}{cz+d}$ such that for all t we have :

$$F(t) = \frac{af(t)+b}{cf(t)+d} \qquad (*)$$

Since the homography depends on three parameters, we need to derive three times to get a condition of equality without parameters. If we derive (*) once, we get

$$F'(t) = \frac{(ad-bc)f'(t)}{(cf(t)+d)^2}$$

Taking the logarithmic derivatives of both sides, we get

$$\frac{F''(t)}{F'(t)} = \frac{f''(t)}{f'(t)} - \frac{2cf'(t)}{cf(t)+d}$$

Let us suppose $c \neq 0$ and put $C = \frac{d}{c}$, we get

$$\frac{F''(t)}{F'(t)} = \frac{f''(t)}{f'(t)} - \frac{2f'(t)}{f(t)+C}$$

or

$$f(t) + C = 2f'(t)\frac{f'(t)F'(t)}{f''(t)F'(t) - F''(t)f'(t)}$$

Deriving one more time we get rid of the constant C and

$$f'(t) = \frac{(f''(t)F'(t) - F''(t)f'(t))(4f'(t)f''(t)F'(t) + 2f'(t)^2F''(t)) - 2f'(t)^2F'(t)(f'''(t)F'(t) - F'''(t)f'(t))}{(f''(t)F'(t) - F''(t)f'(t))^2}$$

Simplifying by $f'(t)$ we get

$$(f''F' - F''f')^2 = 4f''^2F'^2 + 2f'f''F'F'' - 4f'f''F'F'' - 2f'^2F''^2 - 2f'f'''F'^2 + 2f'^2F'F'''$$

or

$$3f''^2F'^2 - 3f'^2F''^2 - 2f'f'''F'^2 + 2f'^2F'F''' = 0$$

Dividing by $2f'^2F'^2$, we get

$$\frac{F'''(t)}{F'(t)} - \frac{3}{2}\frac{F''(t)^2}{F'(t)^2} = \frac{f'''(t)}{f'(t)} - \frac{3}{2}\frac{f''(t)^2}{f'(t)^2}$$

Definition. The *Schwarzian derivative* of a real function of a real variable f of class C^3 in a point t such that $f'(t) \neq 0$, denoted $\{f\}_t$ is :

$$\{f\}_t = \frac{f'''(t)}{f'(t)} - \frac{3}{2}\frac{f''^2}{f'^2}$$

We may also call the function $t \mapsto \{f\}_t$ the *projective acceleration* of the mobile point with abscissa $f(t)$.

From our computations we deduce the following theorem.

Theorem. Two parametric linear curves are projectively equal if and only if they have the same projective acceleration.

Corollary. Let $t \longmapsto K(t)$ be a given function. The equation of order 3

$$\{f\}_t = K(t)$$

defines all the parametric curves with projective acceleration equal to K. If one knows one solution f_0 all the other solution are $t \longmapsto \frac{af_0(t)+b}{cf_0(t)+d}$ where a, b, c and d are real numbers such that $ad - bc \neq 0$.

§ 2. An other method. Normal coordinates

2.1 Parametric linear curve described with homogeneous coordinates

Following the general description of a point on a projective line we may use homogeneous coordinates (x, y) instead of the inhomogeneous coordinate z. The two types of coordinates verify :

$$z = \frac{x}{y}$$

We may describe the parametric curve by a couple of equations

$$\begin{cases} x &= x(t) \\ y &= y(t) \end{cases}$$

2.2 Linear differential equation satisfied by two given functions

Let $x(t)$ and $y(t)$ be two functionsof class C^2. We suppose that the mobile point is realy moving, that is $\frac{x(t)}{y(t)}$ is not a constant or $x(t)y'(t) - x'(t)y(t) \neq 0$.

We consider the second order linear differential equation in the unknown function θ

$$\begin{vmatrix} \theta'' & \theta' & \theta \\ x'' & x' & x \\ y'' & y' & y \end{vmatrix} = 0$$

The solutions of this equation are

$$C_1x + C_2y \quad \text{where } C_1 \in \mathbb{R} \text{ and } C_2 \in \mathbb{R}$$

The equation may be written explicitely

$$(1) \qquad \theta'' + p(t)\theta' + q(t)\theta = 0$$

where $p(t) = -\dfrac{x''y - y''x}{x'y - y'x}$ and $q(t) = \dfrac{x''y' - y''x'}{x'y - y'x}$.

2.3 Some parametric curves equal to the given curve

Let us consider two linearly independant solutions $x_1(t)$ and $y_1(t)$. We have four constants a, b, c and d such that

$$\begin{cases} x_1(t) &= ax(t) + by(t) \\ y_1(t) &= cx(t) + dy(t) \end{cases} \quad \text{with } ad - bc \neq 0$$

The function $z_1(t) = \frac{x_1(t)}{y_1(t)}$ is a homographic function of $z = \frac{x}{y}$. Thus z_1 describes a parametric curve projectively equal to that described by z. But we do not get all the parametric curves projectively equal to that described by z : we could start with $(\mu(t)x(t), \mu(t)y(t))$ for any function μ which never takes the value 0.

2.4 Normal coordinates

Instead of the functions $x(t)$ and $y(t)$ above we may start with $\frac{1}{\lambda(t)}x(t)$ and $\frac{1}{\lambda(t)}y(t)$. Thus the equation replacing (1) shoud give solutions $\theta_1 = \frac{1}{\lambda}\theta$ or

$$\theta = \lambda\theta_1$$

Then

$$\theta' = \lambda'\theta_1 + \lambda\theta_1'$$

and

$$\theta'' = \lambda''\theta_1 + 2\lambda'\theta_1' + \lambda\theta_1''$$

The equation (1) becomes

$$\lambda\theta_1'' + (2\lambda' + p\lambda)\theta_1' + (\lambda'' + p\lambda' + q\lambda)\theta_1 = 0$$

or

$$\theta_1'' + (2\frac{\lambda'}{\lambda} + p)\theta_1' + (\frac{\lambda''}{\lambda} + p\frac{\lambda'}{\lambda} + q)\theta_1 = 0$$

Let us choose λ such that

$$2\frac{\lambda'}{\lambda} + p = 0$$

We have to replace (1) by

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}t^2} + r(t)\theta_1 = 0 \text{ where } r = -\frac{1}{4}p^2 - \frac{1}{2}p' + q$$

Let us skip the index 1. We call *normal* the homogeneous coordinates (x, y) such that x and y are linearly independent solutions of the equation

$$\text{(II)} \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + r(t)\theta = 0$$