

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Proof of L'Hospital's Rule We are assuming that $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. Let

$$L = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \rightarrow a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \quad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I . Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \rightarrow a^+$, then $y \rightarrow a^+$ (since $a < y < x$), so

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{F(x)}{G(x)} = \lim_{y \rightarrow a^+} \frac{F'(y)}{G'(y)} = \lim_{y \rightarrow a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L . Therefore

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.