DETERMINANTS OF VECTORS AND MATRICES

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Preface

Determinants are important tools in analysing and solving systems of linear equations. We have for instance:

Theorem. A system of n linear equations in n unknowns $(x_1, x_2, \ldots, x_n)$

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}\right.$$

has a unique solution if and only if the determinant of the coefficient matrix

$$A = (a_{ij})_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

denoted by $\det A$, $\det(A)$ or $|A|$, is different from zero:

$$\det A \neq 0.$$

1 Some words about linear algebra

1.1 Linear functions (or maps)

A map $f : E \to F$, from a set E to a set F is said to be linear if for any $\vec{u}$ and $\vec{v}$ in E:

$$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$$

But then we must also have $f(\vec{u} + \vec{u}) = f(\vec{u}) + f(\vec{u})$ or $f(2\vec{u}) = 2f(\vec{u})$ and in the same way

$$f(3\vec{u}) = f(\vec{u} + 2\vec{u}) = f(\vec{u}) + f(2\vec{u}) = f(\vec{u}) + 2f(\vec{u}) = 3f(\vec{u}),$$

and so on.

Also, since $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{u} = \vec{u}$, we have

$$f(\tfrac{1}{2}\vec{u}) + f(\tfrac{1}{2}\vec{u}) = f(\tfrac{1}{2}\vec{u} + \tfrac{1}{2}\vec{u}) = f(\vec{u})$$

or $2f(\frac{1}{2}\vec{u}) = f(\vec{u})$ or even better $f(\frac{1}{2}\vec{u}) = \frac{1}{2}f(\vec{u})$.

In that way we get for any rational number $\frac{p}{q}$ the relation $f(\frac{p}{q}\vec{u}) = \frac{p}{q}f(\vec{u})$.

Since every real number λ is a limit of rational numbers, we get the general rule

$$f(\lambda\vec{u}) = \lambda f(\vec{u}).$$

Now we have a better definition of a linear map or function:

A function $f : E \to F$ is linear if for any $\vec{u}$ and $\vec{v}$ in E and any real λ:

$$\begin{cases} f(\vec{u} + \vec{v}) & = & f(\vec{u}) + f(\vec{v}) \\ f(\lambda\vec{u}) & = & \lambda f(\vec{u}) \end{cases}$$

1.2 Linear spaces

Our previous definition of a linear map or function is not yet quite good since we cannot be sure that the addition and the multiplication by a real number have any meaning in the sets E and F. We have to assume that these operations have meaning in the sets E and F, that is to say we must have a specific structure on these sets. The structure we need is called ”linear space”. More precisely:

Definition 1.2.1 A set E is called a *linear space* or a *vector space* (the elements of E will be called elements *or vectors*), if there are two operations defined on E:

the addition $+$ such that:

> $\forall\, \vec{u}, \vec{v}$ and $\vec{w}$: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
>
> $\exists\, \vec{0}$ such that $\forall\, \vec{u}$: $\vec{u} + \vec{0} = \vec{u}$
>
> $\forall\, \vec{u},\ \exists$ an opposite $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$
>
> $\forall\, \vec{u}$ and $\vec{v}$: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

the multiplication of a vector by a real number is such that for any vectors $\vec{u}$ and $\vec{v}$ and for any real numbers λ and μ:

$1\vec{u} = \vec{u}$ and $0\vec{u} = \vec{0}$ and $(-1)\vec{u} = -\vec{u}$

$\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$

$(\lambda + \mu)\vec{u} = \lambda\vec{u} + \mu\vec{u}$

$\lambda(\mu\vec{u}) = (\lambda\mu)\vec{u}$

Example 1.2.2 $\boxed{n = 0}$ A set with only the null vector $\vec{0}$ is a vector space if we put the rules $\vec{0} + \vec{0} = \vec{0}$ and $\lambda\vec{0} = \vec{0}$.

Example 1.2.3 $\boxed{n = 1}$ The set of real numbers is a vector space with the usual addition and multiplication.

Example 1.2.4 $\boxed{n = 2}$ The set of matrices with 1 column and 2 rows

$$\mathbb{R}^{2\times 1} = M_{2,1}(\mathbb{R}) = \left\{ \begin{pmatrix} a \\ c \end{pmatrix} \middle| \; a, c \in \mathbb{R} \right\}$$

becomes a vector space if we define addition and multiplication by a real number in the following "natural" way: for any two vectors in $\mathbb{R}^{2\times 1}$:

$$\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} := \begin{pmatrix} a + b \\ c + d \end{pmatrix},$$

and for any vector in $\mathbb{R}^{2\times 1}$ and for any real number λ:

$$\lambda \begin{pmatrix} a \\ c \end{pmatrix} := \begin{pmatrix} \lambda a \\ \lambda c \end{pmatrix}.$$

The nul vector is then

$$\vec{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 1.2.5 $\boxed{n}$ The set $\mathbb{R}^n$ or $M_{n,1}(\mathbb{R})$ is a vector space.

Example 1.2.6 For any positive integers m and n, the set of matrices $\mathbb{R}^{m\times n} = M_{n,m}(\mathbb{R})$ with n rows and m columns is a vector space.

Example 1.2.7 The set of all real functions defined on $\mathbb{R}$ is a vector space if

$f + g$ is the function $x \mapsto f(x) + g(x)$ and

λf is the function $x \mapsto \lambda f(x)$.

1.3 Examples of linear maps

a) $f : R \to R$

Proposition 1.3.1 A function f from $\mathbb{R}$ to $\mathbb{R}$ is linear if and only if there is a real number a such that for any real number u we have:

$$f(u) = au$$

Proof. If there is such an a, we have

$$\begin{aligned} f(u+v) &= a(u+v) = au + av = f(u) + f(v), \\ f(\lambda u) &= a(\lambda u) = (a\lambda)u = (\lambda a)u = \lambda(au) = \lambda f(u). \end{aligned}$$

Conversely, suppose f is linear. Since $u = 1u = u1$, we get $f(u) = f(u1) = uf(1) = f(1)u$. Let us put $a := f(1)$, we have for any u in $\mathbb{R}$: $f(u) = au$. □

b) $f : \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$

Proposition 1.3.2 A function f from $\mathbb{R}^{2\times 1}$ to $\mathbb{R}^{2\times 1}$ is linear if and only if there is a 2 by 2 matrix A such that for any vector $\vec{u}$ we have:

$$f(\vec{u}) = A\vec{u}.$$

Proof. First suppose there is such a matrix A. Let us write explicitly A and two vectors $\vec{u}$ and $\vec{v}$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then we have

$$f(\vec{u}) = f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} au_1 + bu_2 \\ cu_1 + du_2 \end{pmatrix}.$$

We can check that $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ explicitly by computing both sides of that equality:

$$f(\vec{u}+\vec{v}) = f\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = f\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} a(u_1+v_1)+b(u_2+v_2) \\ c(u_1+v_1)+d(u_2+v_2) \end{pmatrix} = \begin{pmatrix} au_1+bu_2+av_1+bv_2 \\ cu_1+du_2+cv_1+dv_2 \end{pmatrix}$$

and on the other hand:

$$f(\vec{u})+f(\vec{v}) = f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + f\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} au_1+bu_2 \\ cu_1+du_2 \end{pmatrix} + \begin{pmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{pmatrix} = \begin{pmatrix} au_1+bu_2+av_1+bv_2 \\ cu_1+du_2+cv_1+dv_2 \end{pmatrix}$$

We have also to check that for any real λ and any vector $\vec{u}$, we have $f(\lambda\vec{u}) = \lambda f(\vec{u})$: the left side of the equality is in fact

$$f(\lambda\vec{u}) = f\left(\lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = f\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} = \begin{pmatrix} a\lambda u_1+b\lambda u_2 \\ c\lambda u_1+d\lambda u_2 \end{pmatrix} = \begin{pmatrix} \lambda(au_1+bu_2) \\ \lambda(cu_1+du_2) \end{pmatrix},$$

and the right hand side is:

$$\lambda f(\vec{u}) = \lambda f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda\begin{pmatrix} au_1+bu_2 \\ cu_1+du_2 \end{pmatrix} = \begin{pmatrix} \lambda(au_1+bu_2) \\ \lambda(cu_1+du_2) \end{pmatrix},$$

Conversely, if f is linear, let us define

$$\begin{pmatrix} a \\ c \end{pmatrix} := f\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} b \\ d \end{pmatrix} := f\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we can write for any vector $\vec{u}$:

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and thus by linearity

$$\begin{aligned} f(\vec{u}) &= f\left(u_1\begin{pmatrix}1\\0\end{pmatrix}+u_2\begin{pmatrix}0\\1\end{pmatrix}\right)=f\left(u_1\begin{pmatrix}1\\0\end{pmatrix}\right)+f\left(u_2\begin{pmatrix}0\\1\end{pmatrix}\right)=u_1 f\begin{pmatrix}1\\0\end{pmatrix}+u_2 f\begin{pmatrix}0\\1\end{pmatrix} \\ &= u_1\begin{pmatrix}a\\c\end{pmatrix}+u_2\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}u_1a\\u_1c\end{pmatrix}+\begin{pmatrix}u_2b\\u_2d\end{pmatrix}=\begin{pmatrix}u_1a+u_2b\\u_1c+u_2d\end{pmatrix}=\begin{pmatrix}au_1+bu_2\\cu_1+du_2\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}\end{aligned}$$

□

c) $f:\mathbb{R}^{2\times1}\to\mathbb{R}$

Proposition 1.3.3 A function f from $\mathbb{R}^{2\times1}$ to $\mathbb{R}$ is linear if and only if there is a 1 by 2 matrix $(a\ \ b)$ such that for any vector $\vec{u}$ we have:

$$f(\vec{u})=\begin{pmatrix}a&b\end{pmatrix}\vec{u}$$

Proof. Left as an exercise. □

1.4 Dimension and basis

a) Affine plane versus vector plane

A usual plane such as the blackboard or a sheet of paper gives the image of an affine plane in which all the elements, called points, play the same role. To get a vector plane we need to have one element selected. That element will be the null vector $\vec{0}$. In a vector plane the elements are called *vectors*: we can add them using the parallelogram rule. We can also multiply a vector by a number.

From now on, we will only consider the vector plane. Two vectors $\vec{u}$ and $\vec{v}$ are called *collinear* or *parallel* if $\vec{v}=\lambda\vec{u}$ for some real λ or $\vec{u}=\mu\vec{v}$ for some real μ.

Notice that $\vec{0}$ is collinear to any vector.

b) Basis

To be able to compute anything we need real numbers. To specify every vector we have to choose two vectors which are not collinear, let us call them $\vec{i}$ and $\vec{j}$. Now any vector $\vec{u}$ can be written as $\vec{u}=u_1\vec{i}+u_2\vec{j}$ and the couple of numbers (u_1,u_2) is unique: we say that $(\vec{i},\vec{j})$ is a *basis* of the vector plane. Once we have a basis of the vector plane, we have a bijection preserving the operations between the vector plane and $\mathbb{R}^2$.

$$a\vec{i}+c\vec{j} \qquad \longleftrightarrow \qquad \begin{pmatrix}a\\c\end{pmatrix}$$

Therefore the vector plane is often *identified* with $\mathbb{R}^2$.

Since a basis has exactly 2 vectors we say that the *dimension* of the space is 2, $\dim=2$.

c) Matrix associated to a linear map from the vector plane into itself

Let L be a linear map from the vector plane into the vector plane and suppose we have chosen a basis $(\vec{i}, \vec{j})$. The relation $\vec{v} = L(\vec{u})$ may be written

$$v_1\vec{i} + v_2\vec{j} = L(u_1\vec{i} + u_2\vec{j}).$$

Suppose $L(\vec{i}) = a\vec{i} + c\vec{j}$ and $L(\vec{j}) = b\vec{i} + d\vec{j}$, then:

$$\begin{aligned} v_1\vec{i} + v_2\vec{j} &= L(u_1\vec{i} + u_2\vec{j}) = u_1L(\vec{i}) + u_2L(\vec{j}) \\ &= u_1(a\vec{i} + c\vec{j}) + u_2(b\vec{i} + d\vec{j}) = (au_1 + bu_2)\vec{i} + (cu_1 + du_2)\vec{j} \end{aligned}$$

or

$$\begin{cases} v_1 = au_1 + bu_2 \\ v_2 = cu_1 + du_2 \end{cases}$$

or even better:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

We say that the matrix

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is *associated* to the linear map L relatively to the basis $(\vec{i}, \vec{j})$. If we change the basis we get another matrix A'. Then there is an invertible 2×2 matrix P such that A and A' are related by the equality $A' = P^{-1}AP$.

d) Generalization

Let E be a vector space. A sequence of vectors $(e_1, e_2, \ldots, e_n)$ is a *basis* if every vector u in E can be written in one and only one way as:

$$u = u_1e_1 + u_2e_2 + \ldots + u_ne_n.$$

Since a basis has exactly n vectors we say that the *dimension* of the space is n.

For every linear map $f : E \to E$ there is one square matrix of order n associated to f relatively to the basis $(e_1, e_2, \ldots, e_n)$. If one changes the basis, the matrix is changed following the rule $A' = P^{-1}AP$.

2 Geometrical meanings of determinants

2.1 Determinant of a square matrix

For $2{\times}2$ matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc.$$

The system of 2 linear equations in 2 unknowns (x, y)

$$\left\{ \begin{array}{rcrcl} ax & + & by & = & p \\ cx & + & dy & = & q \end{array} \right.$$

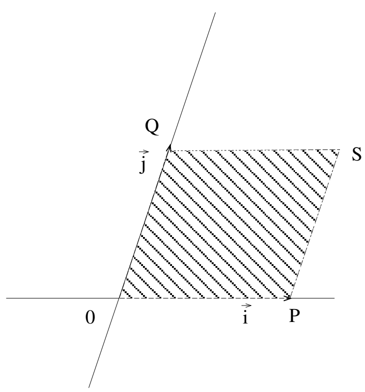
has a unique solution if and only if $ad - bc \neq 0$, and the solution is then

$$\left\{ \begin{array}{rcl} x & = & \dfrac{pd - bq}{ad - bc} \\ y & = & \dfrac{aq - pc}{ad - bc} \end{array} \right.$$

2.2 Area of a parallelogram

Let E be a 2-dimensional vector space with basis $(\vec{i}, \vec{j})$. We choose as unit area the area of the parallelogram constructed on $\vec{i}$ and $\vec{j}$, that is the parallelogram $OPSQ$ such that

$$\overrightarrow{OP} = \vec{i}, \quad \overrightarrow{OS} = \vec{i} + \vec{j} \quad \text{and} \quad \overrightarrow{OQ} = \vec{j}.$$



Problem 2.2.1 Let $\vec{u}$ and $\vec{v}$ be two vectors

$$\vec{u} = a\vec{i} + c\vec{j} \quad \text{and} \quad \vec{v} = b\vec{i} + d\vec{j}.$$

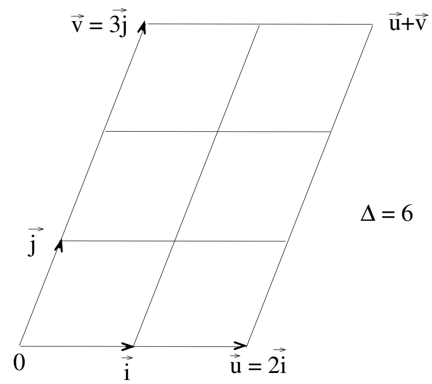
What is the area Δ of the parallelogram constructed on $\vec{u}$ and $\vec{v}$?

To find Δ we need 3 rules:

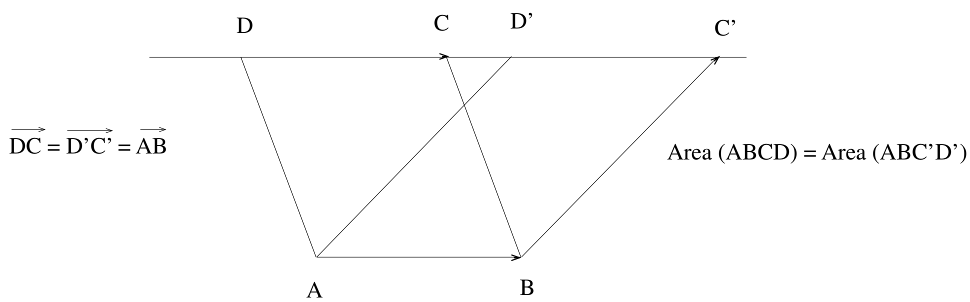
Rule 1. If $\vec{u}$ is parallel to $\vec{i}$ and $\vec{v}$ parallel to $\vec{j}$, that is

$$\vec{u} = a\vec{i} \quad \text{and} \quad \vec{v} = d\vec{j}$$

then $\Delta = ad$.



Rule 2 (Euclid, about 300 BC). The area of a parallelogram does not change when you let one side glide on the line on which it is lying

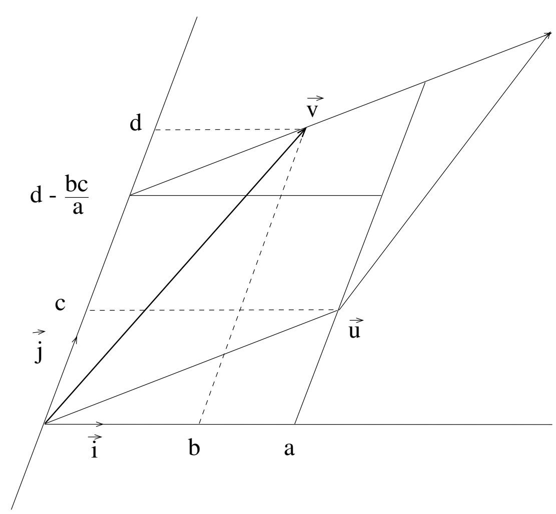


Rule 3. The area is positive if you have to turn in the same direction (to the left or to the right) to move from $\vec{u}$ to $\vec{v}$ as from $\vec{i}$ to $\vec{j}$. The area is negative if you have to turn in opposite directions. Thus the rule 1 is valid even if a and/or d are negative numbers.

Definition 2.2.2 The oriented area Δ is called the determinant of $\vec{u}$ and $\vec{v}$ with respect to the basis $(\vec{i}, \vec{j})$ and denoted by

$$\Delta = \det_{(\vec{i},\vec{j})}(\vec{u}, \vec{v}).$$

2.3 Computation of $\det_{(\vec{i},\vec{j})}(\vec{u}, \vec{v})$ when $\vec{u} = a\vec{i} + c\vec{j}$ and $\vec{v} = b\vec{i} + d\vec{j}$



First, we use rule 2:

$$\begin{aligned}\det_{(\vec{i},\vec{j})}(\vec{u},\vec{v}) &= \det_{(\vec{i},\vec{j})}(\vec{u},\vec{v}-\frac{b}{a}\,\vec{u}) \\ &= \det_{(\vec{i},\vec{j})}\Big(a\vec{i}+c\vec{j},(d-\frac{bc}{a})\vec{j}\Big).\end{aligned}$$

Using rule 2 once again (for the other side) we get

$$\det_{(\vec{i},\vec{j})}(\vec{u},\vec{v}) = \det_{(\vec{i},\vec{j})}\Big(a\vec{i},(d-\frac{bc}{a})\vec{j}\Big).$$

Then the rule 1 gives us

$$\det_{(\vec{i},\vec{j})}(\vec{u},\vec{v}) = a\,(d-\frac{bc}{a})\ \det_{(\vec{i},\vec{j})}(\vec{i},\vec{j}) = ad-bc.$$

The next Theorem gives a characterisation of the function $\det_{(\vec{i},\vec{j})} : E\times E \to \mathbb{R}$.

Theorem 2.3.1 The function $\det_{(\vec{i},\vec{j})} : E\times E \to \mathbb{R}$,

$$(a\vec{i}+c\vec{j},\, b\vec{i}+d\vec{j}) \ \longmapsto\ ad-bc$$

is the only function from $E\times E$ to $\mathbb{R}$ such that

(i) For all $\vec{u}\in E$ the function $\det_{(\vec{i},\vec{j})}(\vec{u},\cdot) : E \to \mathbb{R}$ is linear.

For all $\vec{v}\in E$ the function $\det_{(\vec{i},\vec{j})}(\cdot,\vec{v}) : E \to \mathbb{R}$ is linear.

(ii) For all $\vec{u}\in E$ holds $\det_{(\vec{i},\vec{j})}(\vec{u},\vec{u}) = 0$.

(iii) $\det_{(\vec{i},\vec{j})}(\vec{i},\vec{j}) = 1$.

The (i) means that for any vectors $\vec{u}$, $\vec{v}$, $\vec{u}'$ and $\vec{v}'$, and any numbers λ and μ, we have

$$\begin{aligned}\det_{(\vec{i},\vec{j})}(\vec{u},\lambda\vec{v}+\mu\vec{v}') &= \lambda\det_{(\vec{i},\vec{j})}(\vec{u},\vec{v})+\mu\ \det_{(\vec{i},\vec{j})}(\vec{u},\vec{v}') \\ \det_{(\vec{i},\vec{j})}(\lambda\vec{u}+\mu\vec{u}',\vec{v}) &= \lambda\det_{(\vec{i},\vec{j})}(\vec{u},\vec{v})+\mu\ \det_{(\vec{i},\vec{j})}(\vec{u}',\vec{v}).\end{aligned}$$

Proof. To prove the existence we just have to check that (i), (ii) and (iii) are true. To prove unicity, we proceed in two steps.

Step 1. For any function $\varphi : E\times E \to \mathbb{R}$ such that (i) holds we have for any vectors $\vec{u}$ and $\vec{v}$:

$$\varphi(\vec{u}+\vec{v},\vec{u}+\vec{v}) = \varphi(\vec{u},\vec{u})+\varphi(\vec{u},\vec{v})+\varphi(\vec{v},\vec{u})+\varphi(\vec{v},\vec{v}).$$

If φ is such that (ii) holds, we get

$$0 = 0+\varphi(\vec{u},\vec{v})+\varphi(\vec{v},\vec{u})+0$$

so that

$$\varphi(\vec{v},\vec{u}) = -\varphi(\vec{u},\vec{v}). \tag{1}$$

Step 2. Using (i), we get

$$\varphi(a\vec{i}+c\vec{j},b\vec{i}+d\vec{j}) = ab\,\varphi(\vec{i},\vec{i})+ad\,\varphi(\vec{i},\vec{j})+bc\,\varphi(\vec{j},\vec{i})+cd\,\varphi(\vec{j},\vec{j}).$$

From (ii) and (1) we deduce

$$\varphi(a\vec{i}+c\vec{j},\ b\vec{i}+d\vec{j}) = (ad-bc)\varphi(\vec{i},\vec{j}),$$

and (iii) shows that $\varphi = \det_{(\vec{i},\vec{j})}$. □

2.4 Determinant of a linear function from E to E

Lemma 2.4.1 Let E be a 2-dimensional vector space with a basis $(\vec{i}, \vec{j})$ and let $f : E \to E$ be a linear function. Then there is a number δ_j such that:

$$\forall \vec{u} \in E, \forall \vec{v} \in E \qquad \det_{(\vec{i},\vec{j})}(f(\vec{u}), f(\vec{v})) = \delta_j \det_{(\vec{i},\vec{j})}(\vec{u}, \vec{v}). \tag{2}$$

Proof. Let $\varphi : E \times E \to \mathbb{R}$ be the mapping

$$(\vec{u}, \vec{v}) \longmapsto \varphi(\vec{u}, \vec{v}) = \det_{(\vec{i},\vec{j})}(f(\vec{u}), f(\vec{v})).$$

It easy to check that φ satisfies (i) and (ii) of Theorem 2.3.1. By the same reasoning as above, we get:

$$\varphi(a\vec{i} + c\vec{j}, b\vec{i} + d\vec{j}) = (ad - bc)\,\varphi(\vec{i}, \vec{j}).$$

Thus

$$\delta_f := \varphi(\vec{i}, \vec{j}) = \det_{(\vec{i},\vec{j})}(f(\vec{i}), f(\vec{j}))$$

is such that (2) holds. □

Interpretation of the Lemma

Applying f we transform a parallelogram constructed on any two vectors $\vec{u}$ and $\vec{v}$ into a parallelogram constructed on $f(\vec{u})$ and $f(\vec{v})$. The lemma means that the ratio between the areas of these parallelograms does not depend on the choice of $\vec{u}$ and $\vec{v}$, but only on f. The ratio may be written

$$\delta_f = \det_{(\vec{u},\vec{v})}(f(\vec{u}), f(\vec{v}))$$

for any couple of independent vectors $\vec{u}$ and $\vec{v}$. This justifies the following definition.

Definition 2.4.2 Let $f : E \to E$ be linear. The determinant of f, denoted $\det f$ or $\det(f)$, is the number independent of the choice of the basis $(\vec{i}, \vec{j})$

$$\det f := \det_{(\vec{i},\vec{j})}(f(\vec{i}), f(\vec{j})).$$

Remark 2.4.3 Since $\det f$ is the coefficient which multiplies the areas when we use the transformation f, we have

$$\det(g \circ f) = \det g \cdot \det f.$$

2.5 Two interpretations of the 2×2 matrix determinant

Interpretation 1

Let us denote

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \vec{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the determinant with respect to the standard basis $(\vec{e}_1, \vec{e}_2)$ of the two column vectors of A has the same value as the determinant of the matrix A, provided we preserve their order:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det_{(\vec{e}_1, e_2)}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$$

Interpretation 2

Let $(\vec{i}, \vec{j})$ be a basis of a vector-space E. The matrix A is associated to the linear function $f : E \to E$ is defined by

$$f(\vec{i}) = a\vec{i} + c\vec{j} \quad \text{and} \quad f(\vec{j}) = b\vec{i} + d\vec{j}.$$

Then

$$\det A = \det f.$$

Remark 2.5.1 The relation $\det(g \circ f) = \det g \cdot \det f$ will become

$$\det(BA) = \det B \cdot \det A$$

for all 2×2 matrices B and A.

We may check this formula explicitly. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Then

$$BA = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}$$

and

$$\begin{aligned} \det(BA) &= (a'a + b'c)(c'b + d'd) - (a'b + b'd)(c'a + d'c) \\ &= a'd'ad + b'c'bc - a'd'bc - b'c'ad \\ &= (a'd' - b'c')(ad - bc) = \det B \cdot \det A. \end{aligned}$$

Remark 2.5.2 If we change basis, the matrix associated with f is changed from A to $P^{-1}AP$, where P is an invertible (i.e. regular) matrix and P^{-1} is the inverse of P. We have as we expect

$$\det(P^{-1}AP) = \det A$$

since the two numbers are equal to $\det f$. We can check:

$$\begin{aligned} \det(P^{-1}AP) &= \det P^{-1} \cdot \det A \cdot \det P \\ &= \det P^{-1} \cdot \det P \cdot \det A \\ &= \det(P^{-1}P) \cdot \det A \\ &= \det I \cdot \det A \\ &= \det A. \end{aligned}$$

2.6 Back to the linear system

We write the system

$$\left\{ \begin{array}{ccccc} ax & + & by & = & p \\ cx & + & dy & = & q \end{array} \right.$$

as a linear combination

$$P = xU + yV,$$

where
$$P = \begin{pmatrix} p \\ q \end{pmatrix}, \quad U = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} b \\ d \end{pmatrix}$$

To find x and y, just notice that

$$\det_{(\vec{e}_1,\vec{e}_2)}(P,V) = \det_{(\vec{e}_1,\vec{e}_2)}(xU+yV,V) = x \det_{(\vec{e}_1,\vec{e}_2)}(U,V)$$

and
$$\det_{(\vec{e}_1,\vec{e}_2)}(U,P) = y \det_{(\vec{e}_1,\vec{e}_2)}(U,V)$$

We find the unique solution, whenever $\det_{(\vec{e}_1,\vec{e}_2)}(U,V) = ad - bc \neq 0$, is accordance with Cramer's rule:
$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

2.7 Order 3

Determinant of a square matrix of order 3

We define (or remember that)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Remark 2.7.1 The *Sarrus' rule* is a practical rule for computing determinants of order 3. Use $+$ sign in front of the products of factors on a parallel to the main diagonal of the matrix and $-$ sign for parallels to the other diagonal:

$$\begin{array}{ccccc} & + & & & \\ \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} & a_{11} & a_{12} \\ \ddots & \ddots & \ddots & & \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} & a_{21} & a_{22} \\ & \ddots & \ddots & \ddots & \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} & a_{31} & a_{32} \end{array} \qquad \begin{array}{ccccc} & - & & & \\ \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} & a_{11} & a_{12} \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} & a_{21} & a_{22} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ \boldsymbol{a_{31}} & \boldsymbol{a_{32}} & \boldsymbol{a_{33}} & a_{31} & a_{32} \end{array}$$

2.8 Oriented volume of a parallelepiped constructed on 3 vectors

Suppose that E is a vector space with base $(\vec{i},\vec{j},\vec{k})$ and let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in E. The oriented volume of the parallelepiped constructed on these vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is denoted $\det_{(\vec{i},\vec{j},\vec{k})}(\vec{u},\vec{v},\vec{w})$ if the unit is the volume constructed on the basis.

We may accept the following rules:

Rule 1. $\det_{(\vec{i},\vec{j},\vec{k})}(a\vec{i},b\vec{j},c\vec{k}) = abc$.

Rule 2. The volume of the parallelepiped is not modified when one side is gliding in the plane in which it lies:

$$\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\vec{w}+\lambda\vec{u}+\mu\vec{v}) = \det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\vec{w}).$$

Rule 3. The volume is positive (resp. negative) if $(\vec{u},\vec{v},\vec{w})$ has the same (resp. opposite) orientation as the basis $(\vec{\imath},\vec{\jmath},\vec{k})$.

Computed value of the volume is

$$\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(a_{11}\vec{\imath}+a_{21}\vec{\jmath}+a_{31}\vec{k},a_{12}\vec{\imath}+a_{22}\vec{\jmath}+a_{32}\vec{k},a_{13}\vec{\imath}+a_{23}\vec{\jmath}+a_{33}\vec{k}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Characterisation of the function $\det_{(\vec{\imath},\vec{\jmath},\vec{k})} : E^3 \to \mathbb{R}$

Theorem 2.8.1 The function $\det_{(\vec{\imath},\vec{\jmath},\vec{k})} : (\vec{u},\vec{v},\vec{w}) \longmapsto \det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\vec{w})$ is the only function from E^3 to $\mathbb{R}$ such that

(i) the functions $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\cdot,\vec{v},\vec{w})$, $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\cdot,\vec{w})$ and $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\cdot)$ are linear

(ii) $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{u},\vec{w}) = 0$, $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\vec{u}) = 0$ and $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{u},\vec{v},\vec{v}) = 0$

(iii) $\det_{(\vec{\imath},\vec{\jmath},\vec{k})}(\vec{\imath},\vec{\jmath},\vec{k}) = 1$.

Determinant of a linear function f from E to E

Theorem 2.8.2 Let E be a 3-dimensional vector space and let $f : E \to E$ be linear. The number

$$\det f := \det_{(\vec{\imath},\vec{\jmath},\vec{k})}(f(\vec{\imath}),f(\vec{\jmath}),f(\vec{k}))$$

is independent of the choice of the basis $(\vec{\imath},\vec{\jmath},\vec{k})$.

Definition 2.8.3 The base invariant number $\det f$ is called the *determinant* of the linear function $f : E \to E$ (see Theorem 2.8.2).

Remark 2.8.4 $\det f$ is the multiplication coefficient of volumes when you apply the transformation f, thus

$$\det(g \circ f) = \det g \cdot \det f.$$

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \text{and} \quad \vec{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

you can think of $\det A$ as

$$\det A = \det_{(\vec{e}_1,\vec{e}_3,\vec{e}_3)}\left(\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}\right)$$

or as $\det A = \det f$, if f is a linear function whose associated matrix is A with respect to some basis.

2.9 Solving a system of 3 linear equations in 3 unknowns

Following the idea of Subsection 2.6, let us again write the system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{cases}$$

as a linear combination of the columns:

$$B = x_1V_1 + x_2V_2 + x_3V_3,$$

where

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad V_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad V_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \quad \text{and} \quad V_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}.$$

Notice that

$$\det_{(\vec{e}_1,\vec{e}_3,\vec{e}_3)}(B, V_2, V_3) = x_1 \det_{(\vec{e}_1,\vec{e}_3,\vec{e}_3)}(V_1, V_2, V_3).$$

Thus, if V_1, V_2 and V_3 are linearly independent we again get Cramer's rule

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \; x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \; x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

3 How to compute determinants of matrices

3.1 One inductive rule

Definition 3.1.1 Let A be a square matrix of order $n \geq 2$. We denote by M_{ij}^A and call *minor matrix of A indexed by i and j* when $1 \leq i \leq n$ and $1 \leq j \leq n$, the matrix obtained when you supress the i-row and the j-column of A.

If $A = (a_{ij})_{n \times n}$, then

$$M_{ij}^A = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & & \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

Rule. Let A be a square matrix $A = (a_{ij})_{n \times n}$.

1. If $n = 1$, then $\det(a_{11}) := a_{11}$.

2. If $n \geq 2$, the *cofactor* A_{ij} of the element a_{ij} of A is

$$A_{ij} := (-1)^{i+j} \det(M_{ij}^A).$$

The *determinant* of A is a number $\det A$ such that for any row r and any column k

$$\det A = \sum_{j=1}^{n} a_{rj} A_{rj} = \sum_{i=1}^{n} a_{ik} A_{ik}.$$

Remark 3.1.2 If we use the first formula, we say that we are developing the determinant of A along the row r. If we use the second, we develop along the column k.

Remark 3.1.3 To remember the sign to use, you just have to think of the game chess:

$$\begin{pmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \\ + & - & + & - & + & \\ - & + & - & + & - & \\ + & - & + & - & + & \\ & & \vdots & & & + \end{pmatrix}$$

Example 3.1.4 Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The cofactors are $A_{11} = a_{22}$, $A_{12} = -a_{21}$, $A_{21} = -a_{12}$ and $A_{22} = a_{11}$. There are four possibilities for developing $\det A$, giving happily the same result.

Developing along the first row:

$$\det A = a_{11}A_{11} + a_{12}A_{12} = a_{11}a_{22} - a_{12}a_{21},$$

along the second row:

$$\det A = a_{21}A_{21} + a_{22}A_{22} = -a_{21}a_{12} + a_{22}a_{11},$$

and along the columuns:

$$\begin{array}{rcccl} \det A & = & a_{11}A_{11} + a_{21}A_{21} & = & a_{11}a_{22} - a_{21}a_{12} \\ \det A & = & a_{12}A_{12} + a_{22}A_{22} & = & -a_{12}a_{21} + a_{22}a_{11}. \end{array}$$

Example 3.1.5 Developing along the first row:

$$\begin{vmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \\ -4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ -4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} = 3(-2) + (-2) = -8.$$

Example 3.1.6 Compute $\det A$ for

$$A = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 2 \end{pmatrix}.$$

Let us develop it along the third column (why?). We get

$$\begin{vmatrix} 2 & -1 & 0 & 3 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 1 \\ 0 & -4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & 1 \\ -1 & 2 & 0 \end{vmatrix} = -10 - 12 = -22.$$

3.2 Main properties of determinants

Let $A = (a_{ij})$ be a $n \times n$ square matrix.

1. $\det(AB) = \det A \cdot \det B$.

2. $\det(\lambda I) = \lambda^n,\ \det(\lambda A) = \lambda^n \det A$ for $\lambda \in \mathbb{R}$ (order of A is n).

3. If A is regular, then $\det(A^{-1}) = (\det A)^{-1}$.

4. $\det A^T = \det A$.

5. For triangular (and diagonal) matrices, the determinant is the product of the elements on the main diagonal

$$\begin{vmatrix} \lambda_1 & & & & \\ & \lambda_2 & & * & \\ & & \ddots & & \\ & O & & \lambda_{n-1} & \\ & & & & \lambda_n \end{vmatrix} = \begin{vmatrix} \lambda_1 & & & & \\ & \lambda_2 & & O & \\ & & \ddots & & \\ & * & & \lambda_{n-1} & \\ & & & & \lambda_n \end{vmatrix} = \lambda_1\lambda_2 \ldots \lambda_n.$$

6a. If all the elements of a row are zero, then the determinant is zero.

6b. If all the elements of a column are zero, then the determinant is zero.

7a. If two rows are proportional, the determinant is zero.

7b. If two columns are proportional, the determinant is zero.

8a. If a row is a linear combination of the others, then the determinant is zero.

8b. If a column is a linear combination of the others, then the determinant is zero.

9. Let B be the matrix deduced from A by

(i) permutation of two rows (resp. columns), then $\det B = -\det A$.
(ii) multiplication of all the elements of a row (resp. column) by a number k, then $\det B = k \det B$.
(iii) addition of the multiple of a line (resp, a columns) to an other, then $\det B = \det A$.

10. If P is a regular matrix, then

$$\det(P^{-1}AP) = \det A.$$

3.3 Examples

Example 3.3.1

$$\begin{vmatrix} a & a-1 & a+2 \\ a+2 & a & a-1 \\ a-1 & a+2 & a \end{vmatrix} = \begin{vmatrix} 3a+1 & a-1 & a+2 \\ 3a+1 & a & a-1 \\ 3a+1 & a+2 & a \end{vmatrix} \text{add col 2 and 3 to col 1}$$

$$= (3a+1)\begin{vmatrix} 1 & a-1 & a+2 \\ 1 & a & a-1 \\ 1 & a+2 & a \end{vmatrix} \text{by property 9 (ii)}$$

$$= (3a+1)\begin{vmatrix} 1 & a-1 & a+2 \\ 0 & 1 & -3 \\ 0 & 3 & -2 \end{vmatrix} = (3a+1)\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} = 7(3a+1).$$

Example 3.3.2 Let's compute

$$D_{a,b} = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1-a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1-b \end{vmatrix}.$$

Subtract the second column from the first and after that first row from the result:

$$D_{a,b} = \begin{vmatrix} a & 1 & 1 & 1 \\ a & 1-a & 1 & 1 \\ 0 & 1 & 1+b & 1 \\ 0 & 1 & 1 & 1-b \end{vmatrix} = \begin{vmatrix} a & 1 & 1 & 1 \\ 0 & -a & 0 & 0 \\ 0 & 1 & 1+b & 1 \\ 0 & 1 & 1 & 1-b \end{vmatrix}.$$

Develop along the first column, and then again:

$$D_{a,b} = a\begin{vmatrix} -a & 0 & 0 \\ 1 & 1+b & 1 \\ 1 & 1 & 1-b \end{vmatrix} = a(-a)\begin{vmatrix} 1+b & 1 \\ 1 & 1-b \end{vmatrix} = -a^2(-b^2) = a^2b^2.$$

Example 3.3.3 Compute the determinant of order n

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \cdots & 1 & 1 \\ -1 & -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{vmatrix}.$$

Add the first row to all the others, you get a triangular matrix, with only 1's on the diagonal. Thus $D_n = 1$.

Example 3.3.4 The *Vandermonde* determinant.

Let x_1, x_2,. . ., x_n be n real numbers. Compute the polynomial

$$\Delta(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{vmatrix}.$$

Since $\Delta = 0$, if $x_i = x_j$, we can factorize by $x_i - x_j$. It is easy to see that Δ is homogeneous of degree

$$0 + 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2}.$$

Thus

$$\Delta = k \prod_{i>j} (x_i - x_j)$$

where k is a constant. We determine k by considering the coefficient of $x_2 x_3^2 \ldots x_n^{n-1}$ which is 1. So finally we get

$$\Delta = \prod_{i>j} (x_i - x_j).$$

Example 3.3.5 The *Hilbert* determinant.

Let us compute

$$H_n = \begin{vmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \ldots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \ldots & \frac{1}{2n-1} \end{vmatrix}$$

Substract the last row from the others:

$$H_n = \begin{vmatrix} \frac{n-1}{n} & \frac{n-1}{2(n+1)} & \frac{n-1}{3(n+2)} & \ldots & \frac{n-1}{n(2n-1)} \\ \frac{n-2}{2n} & \frac{n-2}{3(n+1)} & \frac{n-2}{4(n+2)} & \ldots & \frac{n-2}{(n+1)(2n-1)} \\ \frac{n-3}{3n} & \frac{n-3}{4(n+1)} & \frac{n-3}{5(n+2)} & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{1}{(n-1)n} & \frac{1}{n(n+1)} & \frac{1}{(n+1)(n+2)} & \ldots & \frac{1}{(2n-2)(2n-1)} \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \ldots & \frac{1}{2n-1} \end{vmatrix}$$

$$= \frac{(n-1)!}{n(n+1)\dots(2n-1)} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \vdots \\ \vdots & & & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \dots & \frac{1}{2n-2} \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}$$

Substract the last column from the others:

$$\begin{aligned} H_n &= \frac{(n-1)!}{n(n+1)\cdots(2n-1)} \frac{(n-1)!}{n(n+1)\cdots(2n-2)} H_{n-1} \\ &= \prod_{j=1}^{n} \frac{1}{2j-1} \frac{((j-1)!)^4}{((2j-1)!)^2} \end{aligned}$$

We get the following numbers

$$H_1 = 1,\ H_2 = \frac{1}{12},\ H_3 = \frac{1}{180} \cdot \frac{1}{12} = \frac{1}{2160},\ H_4 = \frac{1}{2800} \cdot \frac{1}{2160} = \frac{1}{6048000}, \dots$$

We notice that H_n becomes very small, even compared to the values of the elements of the matrix; it is very useful for testing numerical methods, because this matrix is very unstable with respect to the values of its elements.

4 Definitions and proofs

4.1 Determinant of vectors with respect to a basis

Vocabulary and notations

Definition 4.1.1 We define a number $\varepsilon_{i_1 i_2 \dots i_n}$ depending on its indices as follows.

$\varepsilon_{i_1 i_2 \dots i_n}$ is equal to 0, 1 or -1 according to the following conditions:

$\varepsilon_{i_1 i_2 \dots i_n} = 0$	if two of the indices are equal,
$\varepsilon_{i_1 i_2 \dots i_n} = 1$	if $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ and an <u>even</u> number of transpositions are needed to reorder $i_1\ i_2 \dots i_n$ into $1\ 2 \dots n$,
$\varepsilon_{i_1 i_2 \dots i_n} = -1$	if $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ and an <u>odd</u> number of transpositions are needed to reorder $i_1, i_2 \dots i_n$ into $1\ 2 \dots n$.

Example 4.1.2 $\varepsilon_{12} = 1$, $\varepsilon_{21} = -1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$

$\varepsilon_{112} = \varepsilon_{113} = \varepsilon_{111} = \varepsilon_{212} = \dots = 0$

$\varepsilon_{45132} = \varepsilon_{14532} = -\varepsilon_{12453} = -\varepsilon_{12345} = -1$.

Let S_n be the set of bijections of $\{1, \dots, n\}$ on $\{1, \dots, n\}$, i.e.

$$S_n := \{\, \sigma : \{1, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ bijection} \,\}.$$

S_n is called the *symmetric group of order* n. The elements of S_n are called *permutations*.

If $\sigma \in S_n$, we denote $\sigma_i := \sigma(i)$ for $i = 1, 2, 3, \dots, n$. Note that if $\sigma \in S_n$, then, by bijectivity, $\varepsilon_{\sigma_1 \sigma_2 \dots \sigma_n}$ cannot be zero and belongs to $\{1, -1\}$.

Remember Cartesian product: If E is a set, $E^n := E \times E \times \dots \times E$ is the set of n-uples $(v_1, v_2, \dots, v_n)$, where $v_1 \in E$, $v_2 \in E$, ..., $v_n \in E$.

Definition 4.1.3 Let E be a vector space, and consider a function $\varphi : E^n \to \mathbb{R}$. We use the following terminology:

1. φ is *multilinear* if for all k in $\{1, \dots, n\}$ we have

$$\begin{aligned}\varphi(v_1, \dots, v_{k-1}, \lambda u + \mu w, v_{k+1}, \dots, v_n) &= \lambda \varphi(v_1, \dots, v_{k-1}, u, v_{k+1}, \dots, v_n) \\ &\quad + \mu \varphi(v_1, \dots, v_{k-1}, w, v_{k+1}, \dots, v_n).\end{aligned}$$

2. φ is *alternate* if for all k and h distinct

$$\varphi(v_1, \dots, v_{k-1}, u, v_{k+1}, \dots, v_{h-1}, u, v_{h+1}, \dots, v_n) = 0.$$

3. φ is *antisymmetric* if for all k and h distinct

$$\begin{aligned}&\varphi(v_1, \dots, v_{k-1}, u, v_{k+1}, \dots, v_{h-1}, w, v_{h+1}, \dots, v_n) \\ = \;& -\varphi(v_1, \dots, v_{k-1}, w, v_{k+1}, \dots, u_{h-1}, u, v_{h+1}, \dots, v_n).\end{aligned}$$

Proposition 4.1.4 If φ is multilinear and alternate then it is antisymmetric.

Proof. $0 = \det(\dots, u + v, \dots, u + v, \dots) = \det(\dots, u, \dots, v, \dots) + \det(\dots, v, \dots, u, \dots)$ □

Theorem 4.1.5 Let E be a vector space of dimension n and let $(e_1, \ldots, e_n)$ be a basis of E. For any number λ there is a unique function $\varphi : E^n \to \mathbb{R}$ such that:

(i) φ is multilinear,

(ii) φ is alternate,

(iii) $\varphi(e_1, \ldots, e_n) = \lambda$.

If for $j \in \{1, \ldots, n\}$ we have $v_j = \sum_{i=1}^{n} a_{ij} e_i$, then

$$\varphi(v_1, v_2, \ldots, v_n) = \varphi(e_1, e_2, \ldots, e_n) \sum_{\sigma \in \varphi_n} \varepsilon_{\sigma_1 \sigma_2 \ldots \sigma_n} a_{\sigma_1 1}, a_{\sigma_2 2} \ldots a_{\sigma_n n}.$$

Proof. Let $v_j = \sum_{i=1}^{n} a_{ij} e_i$ for $j = 1, 2, \ldots, n$. Then

$$\varphi(v_1, v_2, \ldots, v_n) = \varphi\left(\sum_{i_1=1}^{n} a_{i_1 1}\, e_{i_1}, \sum_{i_2=1}^{n} a_{i_2 2}\, e_{i_2}, \ldots, \sum_{i_n=1}^{n} a_{i_n n}\, e_{i_n}\right).$$

Let us suppose that φ is multilinear. Then

$$\varphi(v_1, v_2, \ldots, v_n) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \ldots \sum_{i_n=1}^{n} a_{i_1 1} a_{i_2 2} \ldots a_{i_n n} \varphi(e_{i_1}, e_{i_2}, \ldots, e_{i_n}).$$

Let us suppose that φ is also alternate; then $\varphi(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) = 0$ unless there is $\sigma \in S_n$ such that $i_1 = \sigma_1,\ i_2 = \sigma_2, \ldots, i_n = \sigma_n$. And since

$$\varphi(e_{\sigma_1}, e_{\sigma_2}, \ldots, e_{\sigma_n}) = \varepsilon_{\sigma_1 \sigma_2 \ldots \sigma_n} \varphi(e_1, e_1, \ldots, e_n),$$

we may factorize by $\varphi(e_1, e_2, \ldots, e_n)$ getting the last formula of the theorem. If we choose $\varphi(e_1, e_2, \ldots, e_n) = \lambda$, the function φ is determined. To finish the proof we just have to check that the function defined in that way have the properties (i), (ii) and (iii). □

Definition 4.1.6 The function φ such that $\lambda = 1$ is denoted $\det_{(e_1,\ldots,e_n)}$ and the image of $(v_1, \ldots, v_n)$ by that function $\det_{(e_1,\ldots,e_n)}(v_1, \ldots, v_n)$ is called the *determinant of the vectors* $v_1, \ldots, v_n$ with respect to the basis $(e_1, \ldots, e_n)$.

Remark 4.1.7 We may define the hypervolume in the space E of the generalized parallelepiped constructed on $v_1, v_2, \ldots v_n$ as $\det_{(e_1,\ldots,e_n)}(v_1, \ldots, v_n)$ when we take as unit the hypervolume of the generalized parallelepiped constructed on $e_1, \ldots, e_n$.

Corollary 4.1.8 If $(e_1, \ldots, e_n)$ is a basis of the n-dimensional vector space E, and if $\varphi : E^n \to \mathbb{R}$ is multilinear and alternate, then

$$\varphi = \varphi(e_1, \ldots, e_n) \det_{(e_1,\ldots,e_n)},$$

that is to say that for every $(u_1, \ldots, u_n)$ in E^n:

$$\varphi(u_1, \ldots, u_n) = \varphi(e_1, \ldots, e_n) \det_{(e_1,\ldots,e_n)} (u_1, \ldots, u_n).$$

Proof. The assertion follows imediately from the theorem and the definition of $\det_{(e_1,\ldots,e_n)}$. □

Corollary 4.1.9 If $(e_1, \dots, e_n)$ and $(v_1, \dots, v_n)$ are two basis of E, then

$$\det_{(e_1,\dots,e_n)}(v_1, \dots, v_n) \cdot \det_{(v_1,\dots,v_n)}(e_1, \dots, e_n) = 1.$$

Proof. Since $(v_1, \dots, v_n)$ is a basis, $\varphi = \det_{(v_1,\dots,v_n)}$ is multilinear and alternate, thus:

$$\det_{(v_1,\dots,v_n)}(u_1, \dots, u_n) = \det_{(e_1,\dots,e_n)}(u_1, \dots, u_n) \cdot \det_{(v_1,\dots,v_n)}(e_1, \dots, e_n).$$

For $(u_1, \dots, u_n) = (v_1, \dots, v_n)$, we get the expected formula. □

Corollary 4.1.10 Let $(e_1, \dots, e_n)$ be a basis of an n-dimensional vector space E. Then: any n vectors v_1, v_2, …, v_n are linearly independent if and only if $\det_{(e_1,\dots,e_n)}(v_1, \dots, v_n) \neq 0$.

Proof. One side with contradiction: If v_1, v_2, …, v_n are linearly dependent, at least one of the vectors is a linear combination of the others, and since $\det_{(e_1 \dots e_n)}$ is multilinear and alternate $\det_{(e_1,\dots,e_n)}(v_1, \dots, v_n) = 0$.

On the other hand, if v_1, v_2, …, v_n are linearly independent, they form a basis and thus from Corollary 4.1.9 we have $\det_{(e_1,\dots,e_n)}(v_1, \dots, v_n) \neq 0$. □

4.2 Determinant of a linear map from E to E

Theorem and Definition 4.2.1 Let E be a vector space of dimension n and let $f : E \to E$ be linear. The number $\det_{(e_1,\dots,e_n)}(f(e_1), \dots, f(e_n))$ is independent of the basis $(e_1, \dots, e_n)$ of E. This number is called the *determinant* of f and denoted by $\det f$:

$$\det f := \det_{(e_1,\dots,e_n)}(f(e_1), \dots, f(e_n)).$$

Proof. Let $(e_1, \dots, e_n)$ and $(v_1, \dots, v_n)$ be two basis of E. We have

$$\det_{(e_1,\dots,e_n)}(v_1, \dots, v_n) \neq 0. \tag{3}$$

For simplicity, we put $\det_e v := \det_{(e_1,\dots,e_n)}(v_1, \dots, v_n)$.

Since $(e_1, \dots, e_n)$ is a basis and the mapping $E^n \to \mathbb{R}$,

$$(u_1, \dots, u_n) \longmapsto \det_e(f(u_1), \dots, f(u_n)),$$

is multilinear and alternate, the Corollary 4.1.8 gives: for all $(u_1, \dots, u_n) \in E^n$

$$\det_{(e_1,\dots,e_n)}(f(u_1), \dots, f(u_n)) = \det_{(e_1,\dots,e_n)}(f(e_1), \dots, f(e_n)) \cdot \det_{(e_1,\dots,e_n)}(u_1, \dots, u_n).$$

And then for $(u_1, \dots, u_n) = (v_1, \dots, v_n)$:

$$\det_{(e_1,\dots,e_n)}(f(v_1), \dots, f(v_n)) = \det_e v \cdot \det_{(e_1,\dots,e_n)}(f(e_1), \dots, f(e_n)). \tag{4}$$

Since $(v_1, \dots, v_n)$ is a basis and the mapping $E^n \to \mathbb{R}$,

$$(u_1, \dots, u_n) \longmapsto \det_e(u_1, \dots, u_n)$$

is multilinear and alternate, the Corollary 4.1.8 gives: for all $(u_1, \ldots, u_n) \in E^n$

$$\det_{(e_1,\ldots,e_n)}(u_1, \ldots, u_n) = \det_e v \cdot \det_{(v_1,\ldots,v_n)}(u_1, \ldots, u_n).$$

And thus for $(u_1, \ldots, u_n) = (f(v_1), \ldots, f(v_n))$ we have:

$$\det_{(e_1,\ldots,e_n)}(f(v_1), \ldots, f(v_n)) = \det_e v \cdot \det_{(v_1,\ldots,v_n)}(f(v_1), \ldots, f(v_n)) \tag{5}$$

From the equations (3), (4) and (5) we get then:

$$\det_{(e_1,\ldots,e_n)}(f(e_1), \ldots, f(e_n)) = \det_{(v_1,\ldots,v_n)}(f(v_1), \ldots, f(v_n)).$$

□

Corollary 4.2.2 Let $f : E \to E$ be linear. Then $\det f \neq 0$ if and only if f is bijective.

Proof. A linear function f from a linear space of dimension n into a linear space of same dimension, is bijective if and only if it is surjective, that is if and only if the images of basis vectors $f(e_1), \ldots, f(e_n)$ are linearly independent, or

$$\det f = \det_{(e_1,\ldots,e_n)}(f(e_1), \ldots, f(e_n)) \neq 0.$$

□

Proposition 4.2.3 If $f : E \to E$ and $g : E \to E$ are linear, then

$$\det(g \circ f) = \det g \cdot \det f.$$

Proof. If f is not bijective, then $g \circ f$ is not bijective and both sides of the equality are zero.

If f is bijective, $f(e_1), \ldots, f(e_n)$ is a basis for any basis $e_1, \ldots, e_n$ and

$$\begin{aligned}
\det(g \circ f) &= \det_{(e_1,\ldots,e_n)}(g(f(e_1)), \ldots, g(f(e_n))) \\
&= \det_{(f(e_1),\ldots,f(e_n))}(g(f(e_1)), \ldots, g(f(e_n))) \cdot \det_{(e_1,\ldots,e_n)}(f(e_1), \ldots, f(e_n)) \\
&= \det g \cdot \det f.
\end{aligned}$$

□

4.3 Determinant of a square matrix

Definition 4.3.1 Let A be a square matrix of order n. We denote the elements of A by a_{ij} for $1 \leq i \leq n$, $1 \leq j \leq n$. The *determinant* of A, denoted by $\det A$, is the number

$$\det A := \sum_{\sigma \in S_n} \varepsilon_{\sigma_1 \sigma_2 \ldots \sigma_n} a_{\sigma_1 1} a_{\sigma_2 2} \ldots a_{\sigma_n n},$$

where S_n is the set of all permutations of $\{1, 2, 3, \ldots, n\}$.

Proposition 4.3.2 The determinant of a matrix and the determinant of its column vectors with respect to the natural basis $(e_1, \ldots, e_n)$ are the same:

$$\det A = \det_{(e_1,\ldots,e_n)} \left(\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

Proof. A direct consequence of Theorem 4.1.5 and Definition 4.1.6. □

Proposition 4.3.3 Suppose E is a vector space of dimension n. Let $f : E \to E$ be linear and let $(e_1, \ldots, e_n)$ be a basis of E. The matrix A associated with f relatively to the basis $(e_1, \ldots, e_n)$ is such that:

$$\det A = \det f.$$

Proof. The elements a_{ij} are such that $f(e_j) = \sum_{i=1}^n a_{ij} e_i$. Then

$$\det A = \det_{(e_1 \ldots e_n)} (f(e_1), \ldots, f(e_n)) = \det f.$$

□

Corollary 4.3.4 If A and B are square matrix of order n, then

$$\det(AB) = \det A \cdot \det B.$$

Proof. Let A be associated with f, and B with g relatively to some basis. Then

$$\det(BA) = \det(g \circ f) = \det g \cdot \det f = \det B \cdot \det A.$$

□

Proposition 4.3.5 If A is a square matrix then

$$\det A^T = \det A.$$

Proof. Let us write $\operatorname{sgn}(\sigma) := \varepsilon_{\sigma_1 \sigma_2 \ldots \sigma_n}$, when $\sigma \in S_n$. Notice that the function $S_n \to S_n$, $\tau \mapsto \tau^{-1}$, is bijective and $\operatorname{sgn}(\tau) = \operatorname{sgn}(\tau^{-1})$. Thus:

$$\begin{aligned} \det A &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma_1 1} \ldots a_{\sigma_n n} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\, \sigma_1^{-1}} \ldots a_{n\, \sigma_n^{-1}} \\ &= \sum_{\tau \in S_n} \operatorname{sgn}(\tau^{-1})\, a_{1\, \tau_1} \ldots a_{n\, \tau_n} = \sum_{\tau \in S_n} \operatorname{sgn}(\tau)\, a_{1\, \tau_1} \ldots a_{n\, \tau_n} = \det A^T. \end{aligned}$$

An alternate proof is to write $A = PTQ$, with $A^T = Q^T T^T P^T$, where P is a product of Gaussian process matrices with determinant ± 1, Q is a permutation matrix with determinant ± 1 and T triangular. Then

$$\begin{aligned} \det A^T &= \det Q^T T^T P^T = \det Q^T \cdot \det T^T \cdot \det P^T \\ &= \det Q \cdot \det T \cdot \det P = \det P \cdot \det T \cdot \det Q = \det(PTQ) = \det A. \end{aligned}$$

□

4.4 A beautiful formula

Definition and Proposition 4.4.1 Let A be a square matrix of order n. We denote its elements by a_{ij}. The *minor matrix* M_{ij}^A is the matrix obtained by supressing the row i and the column j of the matrix A. The determinant $A_{ij} = \det(M_{ij}^A)$ is called the *cofactor* of the element a_{ij}. The matrix whose element are A_{ij} is the *cofactor matrix*, and the transposed matrix is called the *adjoint matrix* of A and denoted by $\text{adj}\, A$. Thus $(\text{adj}\, A)_{ij} = A_{ji}$. Then:

$$\boxed{A\,\text{adj}\, A = (\text{adj}\, A)A = (\det A)I.}$$

Proof. Since $\det A = \sum_{\sigma \in S_n} \varepsilon_{\sigma_1 \ldots \sigma_n}\, a_{1\,\sigma_1} \cdots a_{n\,\sigma_n}$, we get

$$\det A = \sum_{i=1}^{n} a_{ki} A_{ki} = (A(\text{adj}\, A))_{kk}.$$

If we compute $\sum_{i=1}^{n} a_{ki} A_{ji}$ we get the determinant of a matrix with two equal columns and then $(A(\text{adj}\, A))_{jk} = 0$ if $j \neq k$. Thus the formula. □

Corollary 4.4.2 The matrix A is invertible (regular) if and only if $\det A \neq 0$. If $\det A \neq 0$, then:

$$A^{-1} = \frac{1}{\det A}\,\text{adj}\, A.$$

If $\det A = 0$, A is not invertible, because if it was there would be a matrix B such that $AB = I$, and then $\det A \cdot \det B = 1$, so $\det A = 0$ would be impossible.

4.5 Linear systems

Theorem 4.5.1 Let us consider the following vectors and matrices:

$$V_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, V_2 = \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, V_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix},$$

$$A := \begin{pmatrix} V_1 & \ldots & V_n \end{pmatrix} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},\ B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

The linear system of n equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

has a unique solution if and only if $\det A \neq 0$.

When $\det A \neq 0$, the solution is given for every x_k by

$$x_k = \frac{\det\begin{pmatrix} V_1 & \ldots & V_{k-1} & B & V_{k+1} & \ldots & V_n \end{pmatrix}}{\det A}.$$

Proof. Let us write the system as

$$B = x_1 V_1 + x_2 V_2 + \ldots + x_n V_n.$$

1) If $\det A = 0$, the vectors $V_1, \ldots, V_n$ are linearly dependent and do not generate $\mathbb{R}^n$. If B does not belong to the subspace generated by $V_1, \ldots, V_n$, then the system has no solutions. If B belongs to that subspace then there are scalars ξ_k such that

$$B = \xi_1 V_1 + \ldots + \xi_n V_n,$$

and this means that the vector $\xi := (\xi_1 \ \ldots \xi_n)$ is a solution. We have to show that there are even more solutions. But by linear dependence there is a non-zero vector $\lambda := (\lambda_1 \ \ldots \lambda_n)$ such that

$$\lambda_1 V_1 + \ldots + \lambda_n V_n = \mathbf{0},$$

which implies that also $\xi + \lambda \neq \xi$ is a solution.

2) If $\det A = 0$, the vectors $V_1, \ldots, V_n$ are linearly independent and generate $\mathbb{R}^n$, especially the decomposition $B = \xi_1 V_1 + \ldots + \xi_n V_n$ is unique. Since $\det(V_1 \ \ldots \ V_n)$ is multilinear and alternate, we have

$$\begin{aligned} \det\big(V_1 \quad \ldots \quad V_{k-1} \quad B \quad V_{k+1} \quad \ldots \quad V_n\big) &= x_k \det\big(V_1 \quad \ldots \quad V_{k-1} \quad V_k \quad V_{k+1} \quad \ldots \quad V_n\big) \\ &= x_k \det A. \end{aligned}$$

□