# 1 NORMALIZING THE LEAST-SQUARES SOLUTION OF LINES

## 1.1 Description of the problem

Let $a_1$, $b_1$, $c_1$, $k_1$, $a_2$, $b_2$, $c_2$, $k_2$, $a_3$, $b_3$, $c_3$, $k_3$, be real numbers such that

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = a_3^2 + b_3^2 = 1.$$

We call $d_1$, $d_2$ and $d_3$ the straight lines of equations

$$\begin{cases} k_1a_1x_1 & + & k_1b_1x_2 & = & k_1c_1 \\ k_2a_2x_1 & + & k_2b_2x_2 & = & k_2c_2 \\ k_3a_3x_1 & + & k_3b_3x_2 & = & k_3c_3 \end{cases}$$

Note that in the Java Sketches the scaling numbers $k_1$, $k_2$ and $k_3$ are denoted by $a$, $b$ and $c$.

NOTE FOR ERIC: IN JSXGRAPH FIGURES WE CAN USE SUBSCRIPTS!

The lines form a triangle of vertices $S_1 = d_2 \cap d_3$, $S_2 = d_3 \cap d_1$ and $S_3 = d_1 \cap d_2$, see Figure 1. We define

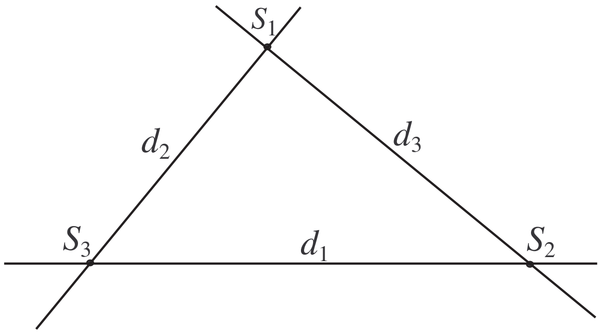



Figure 1: The triangle

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} k_1c_1 \\ k_2c_2 \\ k_3c_3 \end{pmatrix}, \quad A = \begin{pmatrix} k_1a_1 & k_1b_1 \\ k_2a_2 & k_2b_2 \\ k_3a_3 & k_3b_3 \end{pmatrix}$$

The matrix equation $A\mathbf{x} = C$ has in general no solution. The least squares solution $\mathbf{x}_0$ of the normal equation

$$A^T A\mathbf{x} = A^T\mathbf{c}$$

is such that

$$F(\mathbf{x}) = F(x_1, x_2) = \|A\mathbf{x} - \mathbf{c}\|^2 = (\mathbf{x}^T A^T - \mathbf{c}^T)(A\mathbf{x} - \mathbf{c})$$

is minimum for $\mathbf{x} = \mathbf{x}_0$.

We denote by $d(\mathbf{x}, L)$ the distance from the point $\mathbf{x}$ to line $L$. If one computes $F(\mathbf{x})$ one gets:

$$F(\mathbf{x}) = k_1^2 \left(d(\mathbf{x}, d_1)\right)^2 + k_2^2 \left(d(\mathbf{x}, d_2)\right)^2 + k_3^2 \left(d(\mathbf{x}, d_3)\right)^2$$

**I Case** $k_1 = k_2 = k_3$

The function to be minimized is

$$F(\mathbf{x}) = \left(d(\mathbf{x}, d_1)\right)^2 + \left(d(\mathbf{x}, d_2)\right)^2 + \left(d(\mathbf{x}, d_3)\right)^2$$

Let us call $l_1 = S_2S_3$, $l_2 = S_3S_1$ and $l_3 = S_1S_2$, the lengths of the sides of the triangle defined by the lines $d_1$, $d_2$ and $d_3$. The point $\mathbf{x}_0$ such that $F(\mathbf{x})$ is minimum for $\mathbf{x} = \mathbf{x}_0$ is the barycenter of $S_1$, $S_2$ and $S_3$ with masses $l_1^2$, $l_2^2$ and $l_3^2$:

$$\mathbf{x}_0 = \frac{l_1^2}{l_1^2 + l_2^2 + l_3^2} S_1 + \frac{l_2^2}{l_1^2 + l_2^2 + l_3^2} S_2 + \frac{l_3^2}{l_1^2 + l_2^2 + l_3^2} S_3$$

**Problems**

**1.** If the triangle defined by the lines $d_1$, $d_2$ and $d_3$ is isosceles (let us say $S_1S_2 = S_1S_3$), on what line do you expect to find $\mathbf{x}_0$?

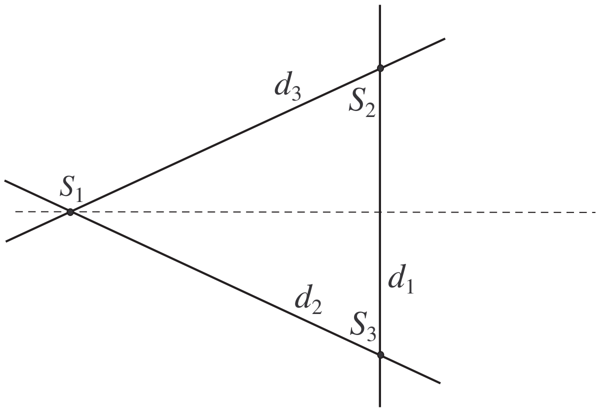



Figure 2: Isosceles triangle

**2.** If the triangle is equilateral where is $\mathbf{x}_0$?

**3.** The distances of $\mathbf{x}_0$ to the sides of the triangle are proportional to the lengths of the corresponding sides to a certain power $r$:

$$\frac{d(\mathbf{x}_0, d_1)}{{l_1}^r} = \frac{d(\mathbf{x}_0, d_2)}{{l_2}^r} = \frac{d(\mathbf{x}_0, d_3)}{{l_3}^r}$$

Guess what is the value of $r$?

Note: If $r = -1$, the point $\mathbf{x}_0$ would be the center of gravity; if $r = 0$, the point $\mathbf{x}_0$ would be the center of the inner circle.

**4.** If the triangle is rectangular, e.g. $d_2 \perp d_3$, on what line do you find $\mathbf{x}_0$ (see Figure 3?

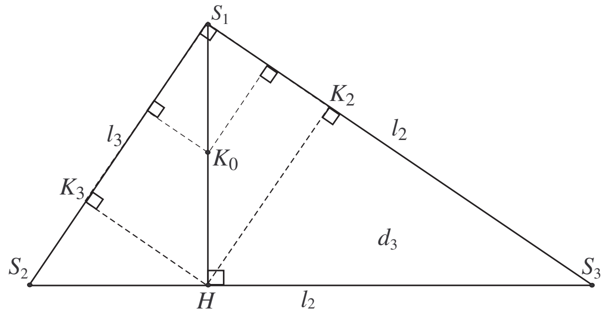



Figure 3: Rectangular triangle

$$\frac{HK_2}{S_1S_3} = \frac{HK_3}{S_2S_1} \text{ since the triangles are similar}$$

**II Case $k_1$, $k_2$ $k_3$ any real numbers**

For this general case we already made experiments!

NOTE FOR ERIC: ONE VERSION OF THE DOCUMENT ENDED HERE

1. Does the triangle $S_1S_2S_3$ change if you change the $k_i$'s ?
2. How does $\hat{\mathbf{x}}$ move if you change $k_1$, keeping all other numbers fixed?
3. Where is $\hat{\mathbf{x}}$ if the triangle is isosceles ($l_2 = l_3$) and $k_2 = k_3$?
4. Where is $\hat{\mathbf{x}}$ if $k_1 = 0$?
5. To a point of what side of the triangle does $\hat{\mathbf{x}}$ tend to, when $k_1 \to \infty$, keeping all other numbers fixed?

   Is it possible to choose $k_2$ and $k_3$ in such a way that $\hat{\mathbf{x}}$ tends to the middle of one side?

   Answer: "Yes, $k_2 = 1/l_2$, $k_3 = 1/l_3$? I'm not sure (??)."
6. If you use a scaling matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

to compute the solution of the scaled least square equation

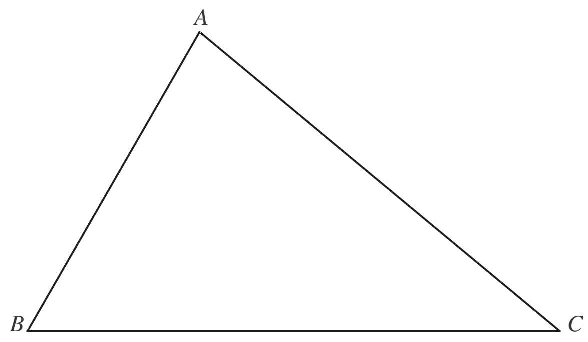
$$A^T D^T D A\mathbf{x} = A^T D^T D\mathbf{c}$$

How should you choose $D$ to get the solution $\hat{\mathbf{x}}$ of the case I?

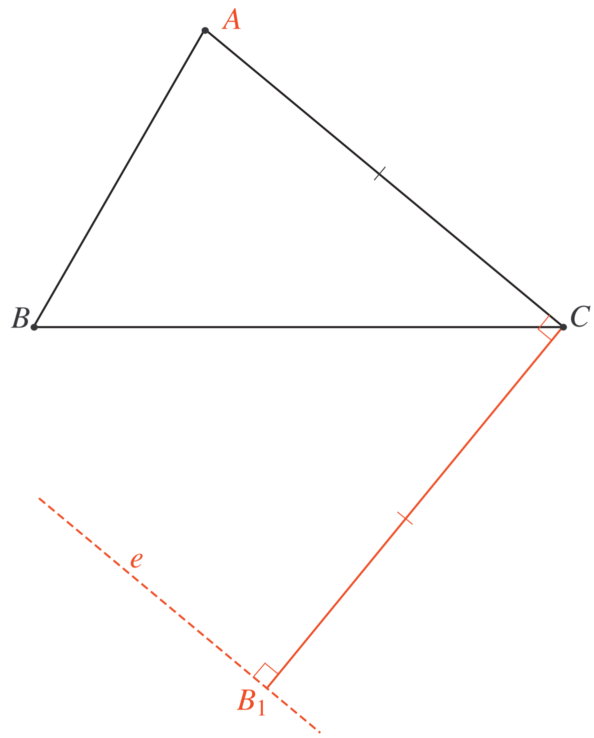
Answer: "$\lambda_i = 1/k_i$."

**Never forget:** always scale before using least squares solution to avoid impact of the choice of units.

## 1.2 A "geometric" construction of x̂ in the case I




1. Construct $B_1$ such that $B_1C = CA$ and $B, C \perp CA$ and $B_1$ and $B$ on the same side of $AC$:



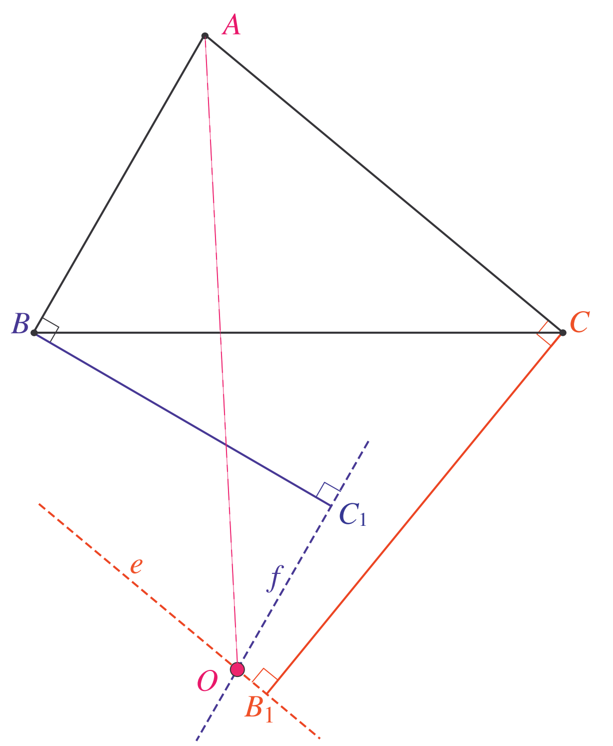

2 Construct the line through $B_1$, parallel to $AC$, call it $e$.

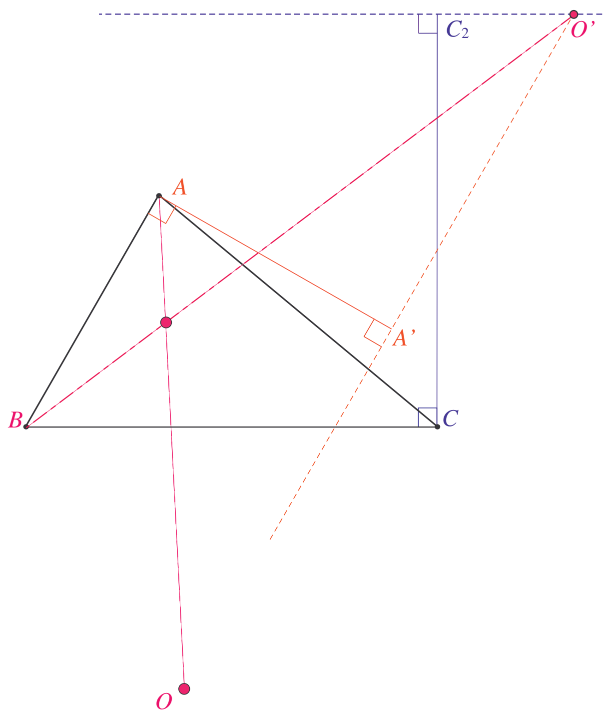
3 Construct $C_1$ such that $BC_1 = BA$, $BC_1 \perp BA$, $C_1$ and $C$ on the same side of $AB$.

4 Construct the line $f$ through $C_1$, parallel to $AB$.

5 Let $O$ be the intersection of $e$ and $f$.

6 Join $AO$. The point x̂ has to be on $AO$.



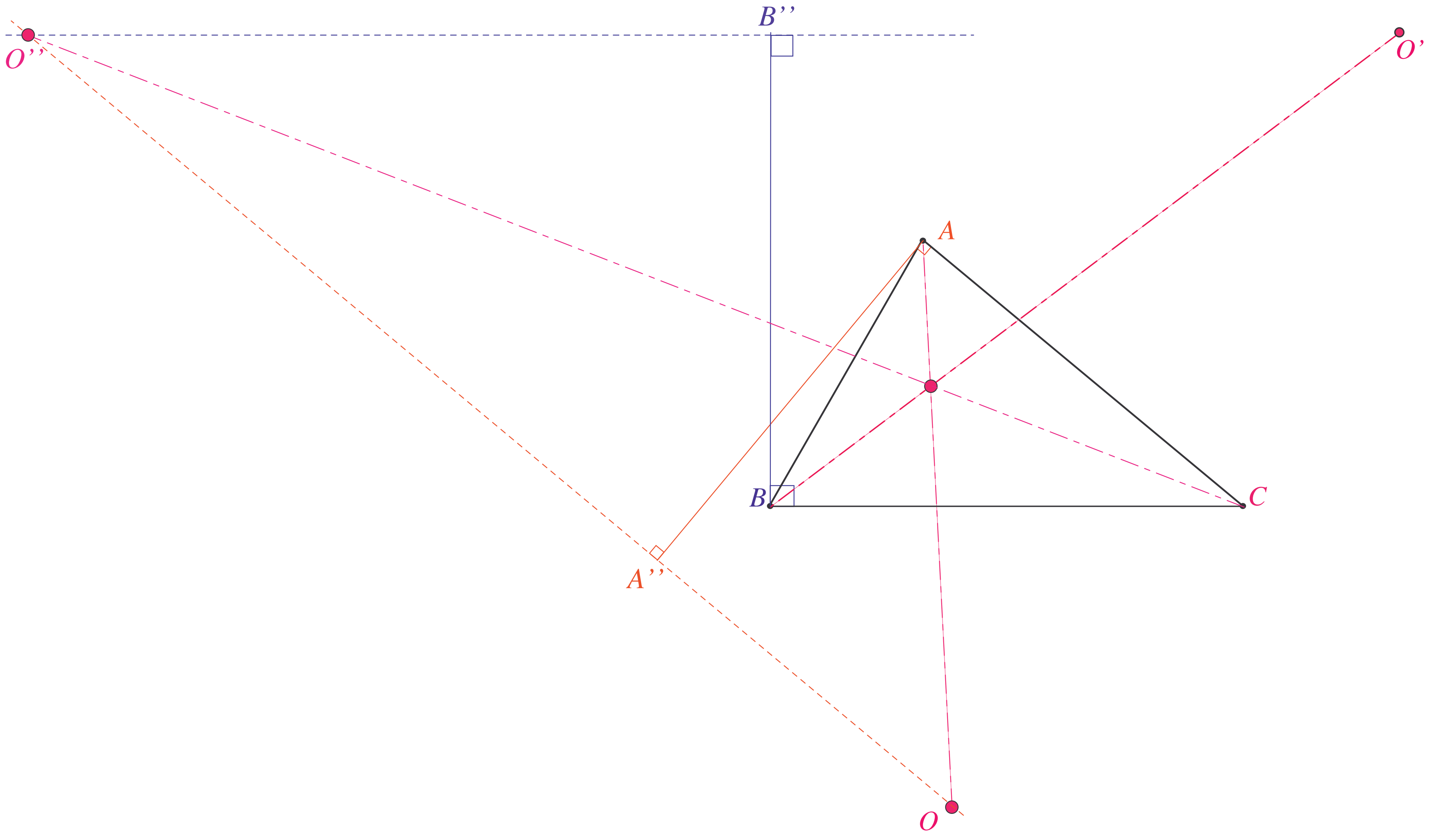




7 Do the same construction from the vertex $B$, getting $BO'$.

8 $\hat{\mathrm{x}}$ is the intersection of $AO$ and $BO'$.

9 Verify with a construction $CO''$ through $C$.

The points $C$, $\hat{\mathrm{x}}$ and $O''$ should be on a straight line.


B''
O''
O'
A
B
C
A''
O