1 NORMALIZING THE LEAST-SQUARES SOLUTION OF LINES

1.1 Description of the problem

Let a_1, b_1, c_1, k_1, a_2, b_2, c_2, k_2, a_3, b_3, c_3, k_3, be real numbers such that

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = a_3^2 + b_3^2 = 1.$$

We call d_1, d_2 and d_3 the straight lines of equations

$$\left\{ \begin{array}{lcccc} k_1 a_1 x_1 & + & k_1 b_1 x_2 & = & k_1 c_1 \\ k_2 a_2 x_1 & + & k_2 b_2 x_2 & = & k_2 c_2 \\ k_3 a_3 x_1 & + & k_3 b_3 x_2 & = & k_3 c_3 \end{array} \right.$$

They form a triangle of vertices $S_1 = d_2 \cap d_3$, $S_2 = d_3 \cap d_1$ and $S_3 = d_1 \cap d_2$, see Figure 1. We

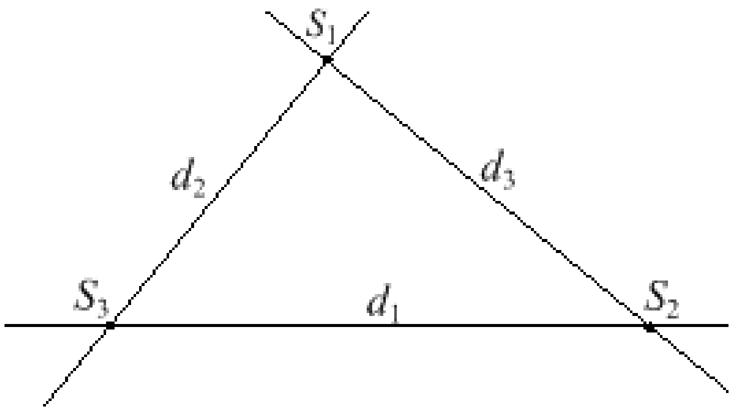


Figure 1: The triangle

define

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} k_1 c_1 \\ k_2 c_2 \\ k_3 c_3 \end{pmatrix}, \quad A = \begin{pmatrix} k_1 a_1 & k_1 b_1 \\ k_2 a_2 & k_2 b_2 \\ k_3 a_3 & k_3 b_3 \end{pmatrix}$$

The matrix equation $A\mathbf{x} = C$ has in general no solution. The least squares solution $\hat{\mathbf{x}}$ of the normal equation

$$A^T A\mathbf{x} = A^T \mathbf{c}$$

is such that

$$F(\mathbf{x}) = F(x_1, x_2) = \|A\mathbf{x} - \mathbf{c}\|^2 = (\mathbf{x}^T A^T - \mathbf{c}^T)(A\mathbf{x} - \mathbf{c})$$

is minimum for $\mathbf{x} = \hat{\mathbf{x}}$.

We denote by $d(\mathbf{x}, L)$ the distance from the point $\mathbf{x}$ to the line L. If one computes $F(\mathbf{x})$ one gets:

$$F(\mathbf{x}) = k_1^2 \left(d(\mathbf{x}, d_1)\right)^2 + k_2^2 \left(d(\mathbf{x}, d_2)\right)^2 + k_3^2 \left(d(\mathbf{x}, d_3)\right)^2$$

I Case $k_1 = k_2 = k_3$

The function to be minimized is

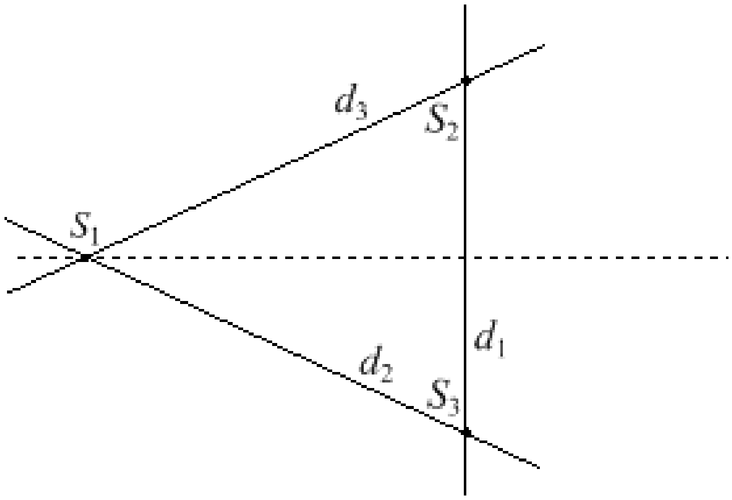
$$F(\mathbf{x}) = (d(\mathbf{x}, d_1))^2 + (d(\mathbf{x}, d_2))^2 + (d(\mathbf{x}, d_3))^2$$

Let us call $l_1 = S_2S_3$, $l_2 = S_3S_1$ and $l_3 = S_1S_2$, the lengths of the sides of the triangle defined by the lines d_1, d_2 and d_3. The point $\hat{\mathbf{x}}$ such that $F(\mathbf{x})$ is minimum for $\mathbf{x} = \hat{\mathbf{x}}$ is the barycenter of S_1, S_2 and S_3 with masses l_1^2, l_2^2 and l_3^2:

$$\hat{\mathbf{x}} = \frac{l_1^2}{l_1^2 + l_2^2 + l_3^2} S_1 + \frac{l_2^2}{l_1^2 + l_2^2 + l_3^2} S_2 + \frac{l_3^2}{l_1^2 + l_2^2 + l_3^2} S_3$$

1.2 Questions

1. If the triangle defined by the lines d_1, d_2 and d_3 is isosceles (let us say $S_1S_2 = S_1S_3$), on what line do you expect to find $\hat{\mathbf{x}}$?



Answer: "$\hat{\mathbf{x}}$ is on the bisector of d_2 and d_3".

2. If the triangle is equilateral where is $\hat{\mathbf{x}}$?

Answer: "$\hat{\mathbf{x}}$ is the (bary)center of the triangle".

3. The distances of $\hat{\mathbf{x}}$ to the sides of the triangle are proportional to the lengths of the corresponding sides to a certain power r:

$$\frac{d(\hat{\mathbf{x}}, d_1)}{l_1{}^r} = \frac{d(\hat{\mathbf{x}}, d_2)}{l_2{}^r} = \frac{d(\hat{\mathbf{x}}, d_3)}{l_3{}^r}$$

What is the value of r?

Note: If $r = -1$, the point $\hat{\mathbf{x}}$ would be the center of gravity; if $r = 0$, the point $\hat{\mathbf{x}}$ would be the center of the inner circle.

Answer: "The value is $r = 1$".

4. If the triangle is rectangular, e.g. $d_2 \perp d_3$, on what line do you find $\hat{\mathbf{x}}$ (see Figure 2?

Answer: "On the height S_1H, where $H \in S_2S_3$ and $S_1H \perp S_2S_3$".

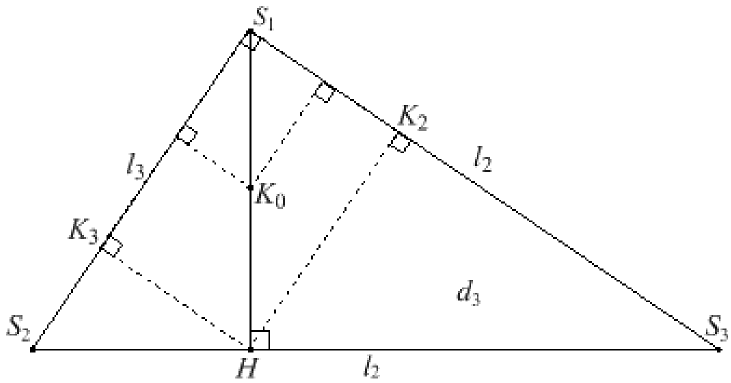


Figure 2: Rectangular triangle

$$\frac{HK_2}{S_1S_3} = \frac{HK_3}{S_2S_1} \text{ since the triangles are similar}$$

II Case k_1, k_2 k_3 any real numbers

1. Does the triangle $S_1S_2S_3$ change if you change the k_i's ?

2. How does $\hat{\mathbf{x}}$ move if you change k_1, keeping all other numbers fixed?

3. Where is $\hat{\mathbf{x}}$ if the triangle is isosceles ($l_2 = l_3$) and $k_2 = k_3$?

4. Where is $\hat{\mathbf{x}}$ if $k_1 = 0$?

5. To a point of what side of the triangle does $\hat{\mathbf{x}}$ tend to, when $k_1 \to \infty$, keeping all other numbers fixed?

Is it possible to choose k_2 and k_3 in such a way that $\hat{\mathbf{x}}$ tends to the middle of one side?

Answer: "Yes, $k_2 = 1/l_2$, $k_3 = 1/l_3$? I'm not sure (??)."

6. If you use a scaling matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

to compute the solution of the scaled least square equation

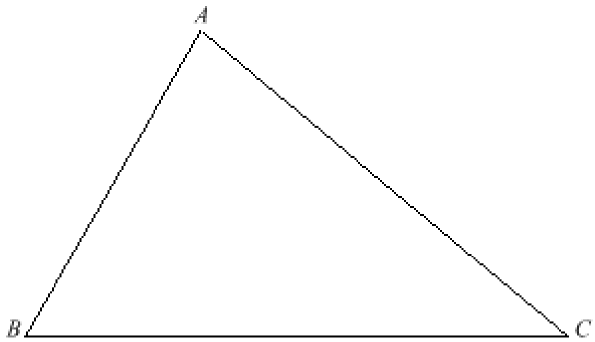
$$A^T D^T D A\mathbf{x} = A^T D^T D\mathbf{c}$$

How should you choose D to get the solution $\hat{\mathbf{x}}$ of the case I?

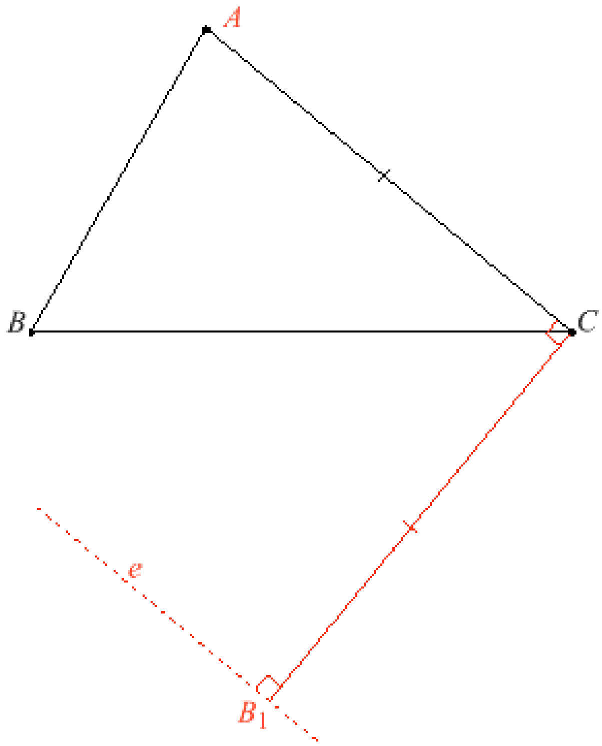
Answer: "$\lambda_i = 1/k_i$."

Never forget: always scale before using least squares solution to avoid impact of the choice of units.

1.3 A "geometric" construction of $\hat{x}$ in the case I



1. Construct B_1 such that $B_1C = CA$ and $B, C \perp CA$ and B_1 and B on the same side of AC:



2 Construct the line through B_1, parallel to AC, call it e.

3 Construct C_1 such that $BC_1 = BA$, $BC_1 \perp BA$, C_1 and C on the same side of AB.

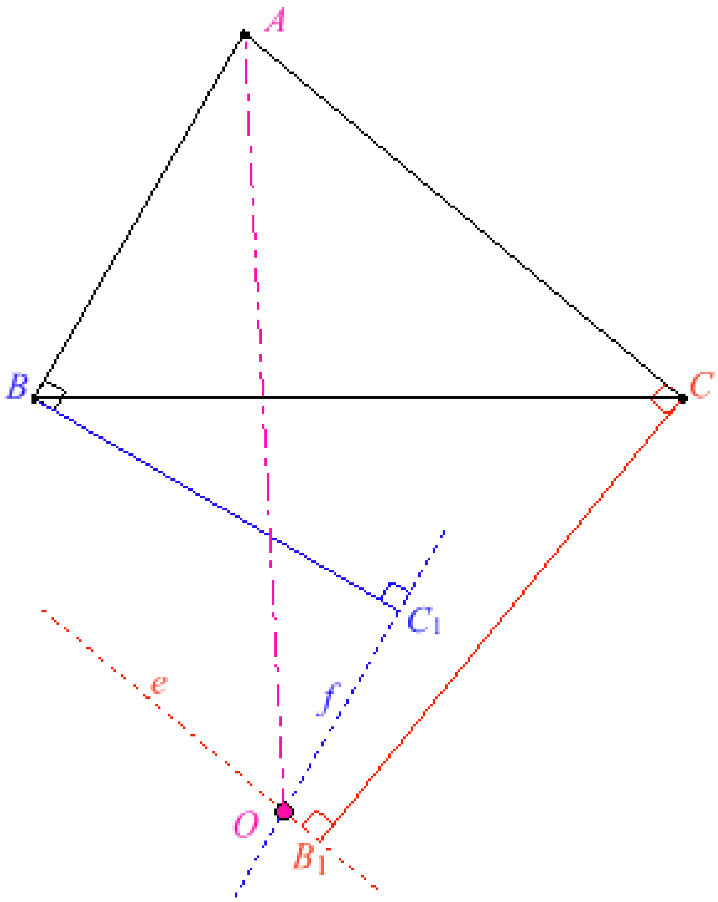
4 Construct the line f through C_1, parallel to AB.

5 Let O be the intersection of e and f.

6 Join AO. The point $\hat{x}$ has to be on AO.

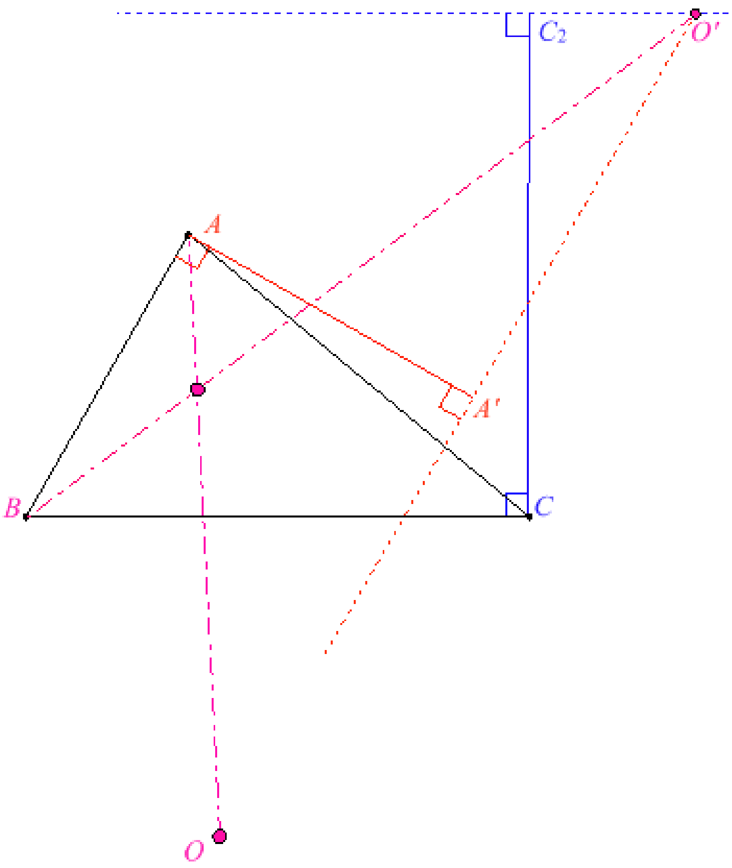
7 Do the same construction from the vertex B, getting BO'.

8 $\hat{x}$ is the intersection of AO and BO'.

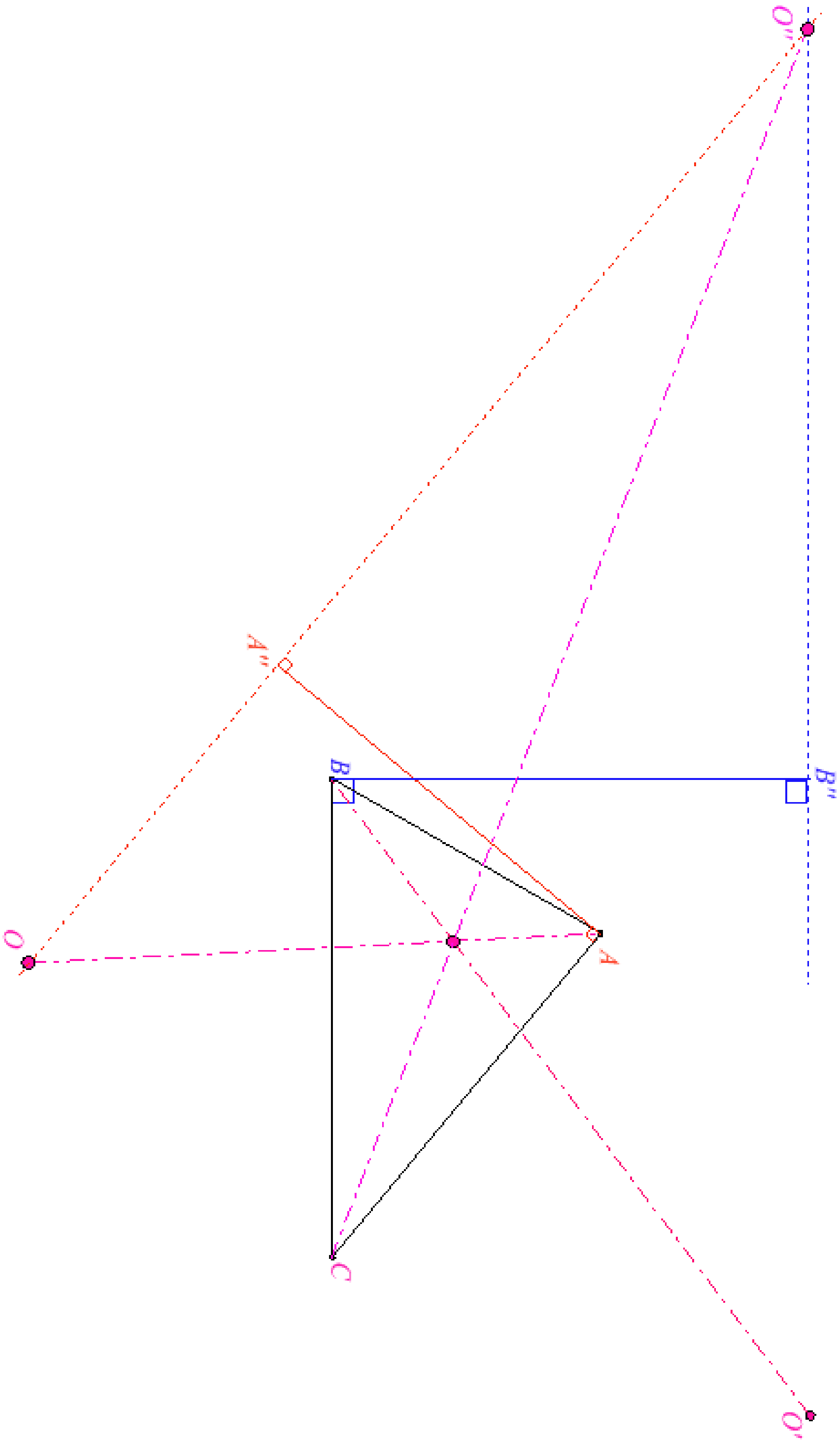


9 Verify with a construction CO'' through C.

The points C, $\hat{\mathrm{x}}$ and O'' should be on a straight line.



C_2
O'
A
A'
B
C
O


O''
A''
B
B''
O
A
C
O'