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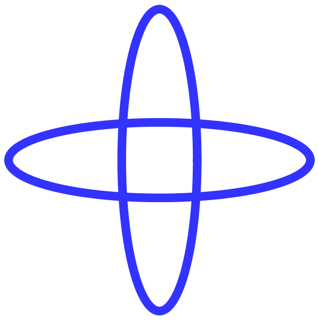

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15 mars 2011

Chapitre 1

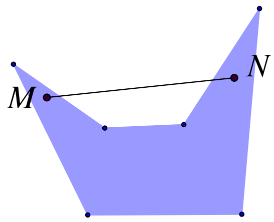
Regular convex polyhedra

1.1 Vocabulary

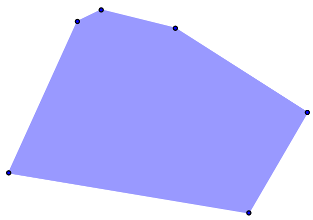
Definition. A subset D of a plane P or of the three-dimensional space E is *convex* if

$$\forall M \in D \quad \forall N \in D \quad (\text{segment } MN) \subset D$$

Counter-examples and examples.



Non convex subset of P



Convex subset of P

A disc is convex, but a ring is not. In space the inside of a sphere, called a ball, is convex. The inside of a cube is convex, but a torus (shape of a doughnut) is not.

Harjoitus 1. Show that the intersection of a family of convex subsets of P (respectively E) is convex.

Proposition and definition. Let A be a subset of a plane P (or of space E). There is a subset H of P (respectively E) which is the smallest (relatively to the inclusion relation) of all convex subsets containing A. This subset H is called the *convex hull* of A.

Proof. The plane P is a convex subset of P containing A. Thus the family of convex subsets of P containing A is not empty. It follows from the exercise above that the intersection H of all the subsets of this family is a convex subset containing A. The subset H is included in all the convex subsets containing A, thus it is the smallest.

Comment. There are plenty of definitions of polygons and polyhedra. We give here definitions for convex polygons and polyhedra, which are easyly generalized to any dimension.

Definition. A *convex polygon* Π is the convex hull of a finite set $\mathcal{M}$ of points in a plane, which are not on one line. Let $\mathcal{V}$ be the smallest subset of $\mathcal{M}$ such that the convex hull of $\mathcal{V}$ is Π. The points belonging to $\mathcal{V}$ are called the *vertices* of Π. Let A_i and A_j be two distinct vertices, the segment A_iA_j is a *diagonal* of Π if there are points belonging to $\mathcal{V}$ on both sides of the line A_iA_j ; the segment A_iA_j is an *edge* or *side* of Π if all the points belonging to $\mathcal{V}$ are in one of the two half-planes bordered by the line A_iA_j.

A *triangle* has 3 vertices, 3 sides or edges and no diagonal.
A *quadrangle* has 4 vertices, 4 sides or edges and 2 diagonals.
A *pentagon* has 5 vertices, 5 sides or edges and 5 diagonals.
An *hexagon* has 6 vertices, 6 sides or edges and 9 diagonals.
An *heptagon* has 7 vertices, 7 sides or edges and 14 diagonals.
An *octogon* has 8 vertices, 8 sides or edges and 20 diagonals.
An *nonagon* has 9 vertices, 9 sides or edges and 27 diagonals.
An *decagon* has 10 vertices, 10 sides or edges and 35 diagonals.

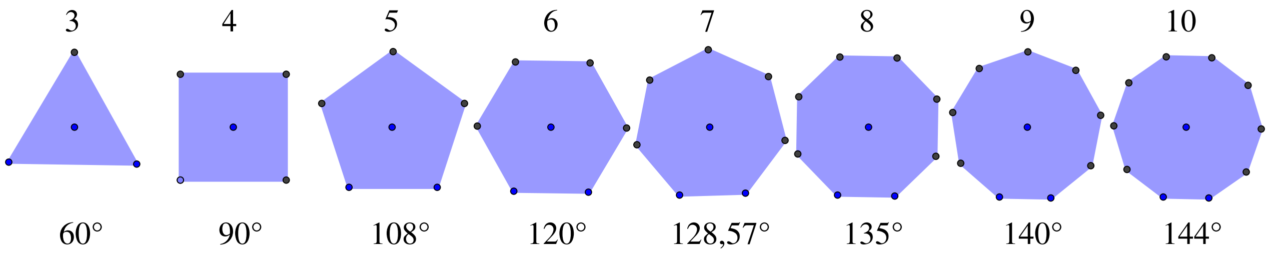
An *n-gon* has n vertices, n sides or edges and $\frac{n(n-3)}{2}$ diagonals.

Notation. The vertices of an n-gon may be denoted by letters indexed from 1 to n : $A_1, A_2, \dots, A_n$ in such a way that the sides are the segments $A_1A_2, \dots, A_{n-1}A_n$ and A_nA_1. We then denote the polygon by the finite sequence of its vertices $\Pi = A_1A_2\dots A_n$.
Definition. A convex polygon $\Pi = A_1A_2\dots A_n$ is *regular* if all its sides are of equal length

$$A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n = A_nA_1$$

and all the angles $\widehat{A_nA_1A_2}, \widehat{A_1A_2A_3}, \dots, \widehat{A_{n-2}A_{n-1}A_n}, \widehat{A_{n-1}A_nA_1}$ have same measure.

$$\theta_n = \widehat{A_nA_1A_2} = \dots = \widehat{A_{n-1}A_nA_1}$$



Definition. A *convex polyhedron* Π is the convex hull of a finite set $\mathcal{M}$ of points in space, which are not on one plane. Let $\mathcal{V}$ be the smallest subset of $\mathcal{M}$ such that the convex hull of $\mathcal{V}$ is Π. The points belonging to $\mathcal{V}$ are called the *vertices* of Π. A *face* of the convex polyhedron Π is the convex hull of a subset $\mathcal{F}$ of $\mathcal{V}$ such that :

1. all the points belonging to $\mathcal{F}$ are in one plane ; we denote it by P ;
2. $\mathcal{F}$ is the set of all the points belonging to $\mathcal{V}$ which are in the plane P ;
3. all the points belonging to $\mathcal{V}$ are in one of the two half-spaces bordered by P.

The sides of the faces are the *edges* of the polyhedron.

Definition. A convex polyhedron Π is *regular* if all its faces are isometric regular convex polygons and each vertex is common to as many faces. It is described by a *Schläfli symbol* $\{p, q\}$ where p is the number of sides of the faces and q is the number of faces at each vertex. For instance the cube has the symbol $\{4, 3\}$.

There are 5 regular convex polyhedra or platonic solids :

1. The *tetrahedron* with Schläfli-symbol $\{3, 3\}$.
2. The *cube* with Schläfli-symbol $\{4, 3\}$.
3. The *octahedron* with Schläfli-symbol $\{3, 4\}$.
4. The *dodecahedron* with Schläfli-symbol $\{5, 3\}$.
5. The *icosahedron* with Schläfli-symbol $\{3, 5\}$.

Harjoitus 2. Sketch the five platonic solids here :

Name	Schläfli-symbol	Faces	F	E	V	$F - E + V$
regular tetrahedron	$\{3, 3\}$	Triangles	4	6	4	?
cube	$\{4, 3\}$	Squares	6	12	8	?
regular octahedron	$\{3, 4\}$	Triangles	8	12	6	?
regular dodecahedron	$\{5, 3\}$	Pentagons	12	30	20	?
regular icosahedron	$\{3, 5\}$	Triangles	20	30	12	?

F is the number of faces, E the number of edges and V the number of vertices.

If you take as vertices of a polyhedron Π' the centers of the faces of a regular convex polyhedra Π you get a regular convex polyhedra called the *dual* of Π. The tetrahedron is his own dual. The cube and the octahedron are dual. The dodecahedron and the icosahedron are dual. If Π has the Schläfli-symbol $\{p, q\}$, the Schläfli-symbol of the dual Π' of Π is $\{q, p\}$.

Harjoitus 3. What is a regular hexahedron ? Answer : a cube !

Harjoitus 4. In a triangle the heights are meeting in a point called the orthocenter. How do you generalize this property to a regular tetrahedron ? Is it true for all tetrahedra ? A tetrahedron is called *orthocentric* if opposite edges are orthogonal ; why ?

1.2 Euler's formula

1.2.1 The theorem

The following definition of a graph is enough for what we want to prove here. More about graphs in chapter 3.

Definitions. A *graph* G is a couple $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set and $\mathcal{E}$ is a subset of all the subsets of $\mathcal{V}$ having 2 elements. The elements of $\mathcal{V}$ are called *vertices* and the elements of $\mathcal{E}$ are called *edges* of G.

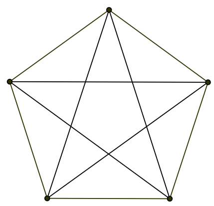
A graph is *connected* if for any two vertices M and N, there is a sequence of vertices A_0, A_1, ..., A_n such that

- for all i from 1 to n the subset $\{A_{i-1}, A_i\}$ is an edge
- $A_0 = M$
- $A_n = N$.

A graph $G = (\mathcal{V}, \mathcal{E})$ is planar if it is possible to represent each vertex M by a point M in a plane P and each $\{M, N\}$ by a continuous line denoted MN with ends M and N in such a way that :

> if two curves MN and $M'N'$ have a point in common than this point is a common endpoint to MN and $M'N'$.

Example and counter-example. The complete graph of order n, called K_5, is defined by $K_n = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ has n elements and $\mathcal{E}$ is the set of all the subsets of $\mathcal{V}$ having 2 elements. Here is a representation of K_5 :



The graph K_4 is planar but K_5 is not.

Harjoitus 5. Draw a picture that shows that K_4 is a planar graph.

Harjoitus 6. Draw pictures representing K_1, K_2, K_3 and K_4. In each case count the number of faces F, the number of edges E and the number of verices V, and verify the Euler formula $F - E + V = 2$. *Hint. Do not forget to count the face which is the part of the plane P which is outside of the figure.*

Euler's formula

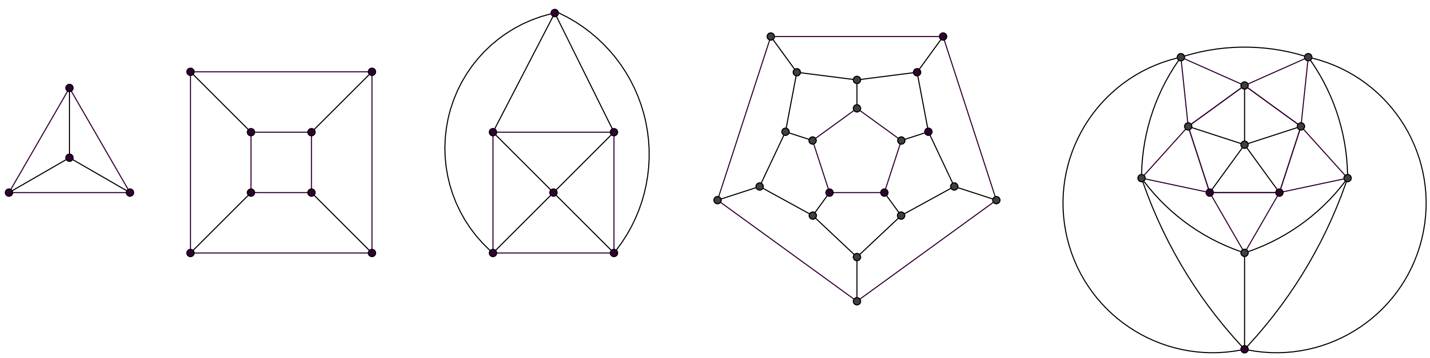
Theorem. For any convex polyhedra

$$\boxed{F - E + V = 2}$$

where F is the number of faces, E the number of edges and V the number of vertices. In fact the formula is valid for any polyhedra whose edges do not project into intersecting arcs on a sphere by central projection.

1.2.2 Proof number 1

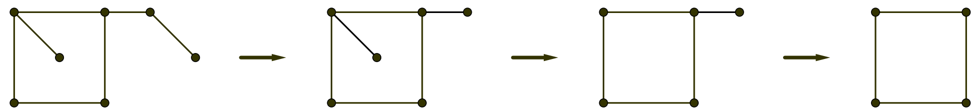
First step. Project the edges of the polyhedra on a sphere by a central projection with projection center at the center of the sphere. The edges transform into arcs of great circles (a *great circle of a sphere* is a circle that has as center the center of the sphere). You may move the vertices on the the sphere without changing the values of F, E and V. Let the vertices move down to come near a "horizontal" plane tangent to the sphere. Finally project from the North pole the figure onto the horizontal plane. You get a connected planar graph and the problem is to show the validity of Euler's formula for a connected planar graph where the exterior of the figure is counting as 1 face. For instance the regular convex polyhedra give



Second step. We proceed recursively step by step until we have reduced the connected planar graph to K_1 that is one vertex. Then we have $F = 1$, $E = 0$ and $V = 1$, so the Euler formula is valid. Procced as follows :

A. If there are vertices which belongs only to one edge (that can happen when some edges have been taken away), take away one of these vertices and simultaneously the corresponding edge. That keeps F and substracts 1 from E and from V : it does not change $F - E + V$. The new graph is still a connected graph. Go on with each of the vertices belonging only to one edge. and then go on to **B.**

For instance, starting from a graph G_1, you transform it to G_2, G_3 and G_4, without changing the number $F - E + V$



G_1 $2-7+7=2$ | G_2 $2-6+6=2$ | G_3 $2-5+5=2$ | G_4 $2-4+4=2$

B. If all the vertices belong to at least 2 edges, take away one edge. This preserves connectedness, substracts 1 from E and 1 from F, while V does not change, so $F - E + V$ does not change. Go to **A**.

C. Since there are only a finite number of vertices and edges, after a finite number of steps, one reaches K_1. □

1.2.3 Proof number 2

This proof begins the same way as the first one, but we stay on the sphere.

A *great circle* of a sphere is a circle included in the sphere of maximum radius, that is the radius of the sphere. The center of a great circle C of a sphere Σ is the center O of the sphere. Given two points M and N of a sphere, a *geodesical arc joining these points* is an arc of a great circle whose endpoints are M and N. If M and N are not opposite points on the sphere, there are 2 geodesical arcs joining M and N. If M and N are opposite points on Σ, then all the half-circles with diameter the segment MN are geodesical arcs joining M and N. There are infinte many of them.

A *spherical triangle* on a sphere Σ is given by three points A, B and C belonging to Σ and three geodesical arcs joining these points two by two, denoted $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ and $\overset{\frown}{AB}$. These geodesical arcs are called the *sides* of the spherical triangle. The angles, measured in radians, of the spherical triangle are the angles done by the half-tangent to the sphere at the endpoints of geodesical arcs which are the sides of it. We suppose that the radius of the sphere is 1. Then we call solid angle defined by the spherical triangle its area. We denote it by $\overset{\triangle}{\overline{ABC}}$.

Examples. The area of the hole sphere is 4π. The area of the triangle NAB, where N is the north pole, A and B points on the equator such that OA and OB are orthogonal is $1/8$th of the hole sphere, that is $\pi/2$.

We show in 1.2.4 "Slices of melon", that the sum of the three angles of a spherical triangle is equal to $\pi +$the area of the spherical triangle :

$$\widehat{BAC} + \widehat{CBA} + \widehat{ACB} = \pi + \overset{\triangle}{\overline{ABC}}$$

First step. Project the edges of the polyhedra on a sphere by a central projection with projection center at the center of the sphere. The edges transform into arcs of great circles. This time you do not move the vertices on the the sphere.

Second step. Add one diagonal as an edge for each quadrilateral : by drawing a diagonal of a quadrilateral, you add 1 to the number E of vertices but also 1 to the number F of faces, the number V of vertices remaining constant ; thus the number $F - E + V$ does not change. Same thing for pentagons : you draw two diagonals from one vertex, adding 2 to E and 2 to F, keeping V constant (you may notice that you could also do that for any planar graph).

As a consequence, we may suppose that our graph contains only trilinear faces.

Third step. If you count the sides of all the triangles, you count each edge twice, thus

$$3F = 2E$$

If you add all the angles of all the triangles you get 2π at each vertex, so the sum is $2\pi V$. But if you count triangle by triangle you get π for each triangle + the area of the triangle. The total sum is thus πF+the total area of the sphere, that is 4π, thus

$$2\pi V = \pi F + 4\pi$$

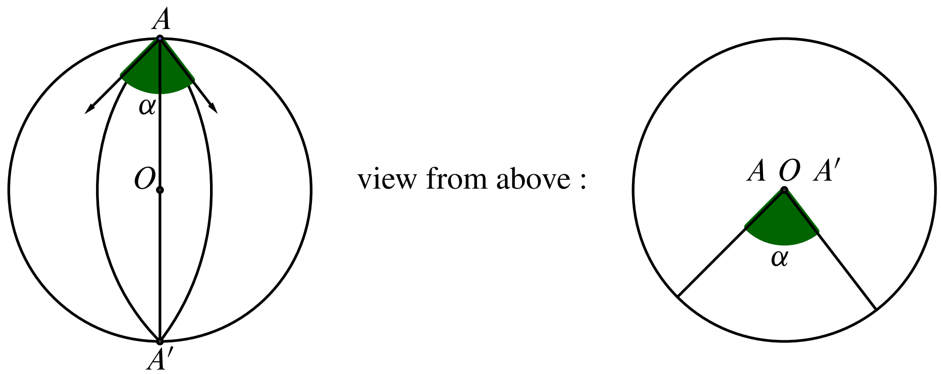
Finally

$$F - E + V = F - \frac{3}{2}F + \frac{1}{2}F + 2 = 2. \qquad \square$$

1.2.4 Slices of melon

We consider a spere Σ of radius 1, with center O.

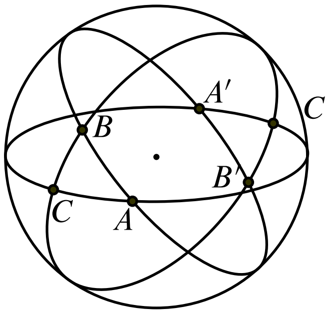
Let A be a point of Σ and let A' be the point opposite to A on Σ. The point A' is the intersection, different from A, of Σ with the line AO. All the great circles going through A are also going through A'. Let us look at two great circles going through A and let α denote the angle of the tangents to these circles at point A.



The area of the melonslice (spherical diedral) is proportional to its angle α. Since for a complete tour one gets the whole sphere of area 4π, corresponding to an angle 2π, the coefficient of proportionality is 2 and

The area of the melon slice with angle α is 2α.

Let's consider a spherical triangle with vertices A, B and C. Denote the opposite points by A', B' and C'. By central symetry, the area of the spherical triangle $A'B'C'$ is the same as the area of the spherical triangle ABC. Let us denote that area by S and the angles by α, β and γ :



We are going to cover the whole sphere with 6 melon slices : those with summits the vertices of the triangle. By doing so we cover the whole sphere once everywhere but on the triangles ABC and $A'B'C'$ which are covered 3 times, that is 2 times to much to get the sphere once. Thus $2 \times 2\alpha + 2 \times 2\beta + 2 \times 2\gamma - 2 \times 2S = 4\pi$ and finally :

$$\alpha + \beta + \gamma = \pi + S \qquad \square$$

1.3 Generalizations

1.3.1 Higher dimensions

Wikipedia says : "In the mid-19th century the Swiss mathematician Ludwig Schläfli discovered the four-dimensional analogues of the Platonic solids, called convex regular 4-polytopes. There are exactly six of these figures ; five are analogous to the Platonic solids, while the sixth one, the 24-cell, has no lower-dimensional analogue. In all dimensions higher than four, there are only three convex regular polytopes : the simplex, the hypercube, and the cross-polytope. In three dimensions, these coincide with the tetrahedron, the cube, and the octahedron." The simplex is auto-dual. The hypercube and the cross-polytope are dual.

Harjoitus 7. Let $(O, \vec{\imath}, \vec{\jmath})$ be an orthonormal frame of a plane. A square may be described by its 4 vertices with coordinates $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$. Let $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$ be an orthonormal frame of a 3-space. How would you describe a cube centered at O with faces parallel to the coordinate planes and length of the edges equal to 2 ? Cut that cube with the plane having the equation $x + y + z = 0$. Show that the section is a regular hexagone. Let us describe a hypercube in an n-dimensional real space by its vertices with coordinates $(\pm 1, \pm 1, \dots, \pm 1)$. Show that the hypercube has 2^n vertices, $n2^{n-1}$ edges, ... , $\binom{n}{k} 2^{n-k}$ k-dimensional faces, ... , $n2$ $(n-1)$-dimensional hyperfaces.

Harjoitus 8. Let $(O, \vec{\imath}, \vec{\jmath})$ be an orthonormal frame of a plane. A square may be described by its 4 vertices with coordinates $(0, 1)$, $(1, 0)$, $(0, -1)$ and $(-1, 0)$. Let $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$ be an orthonormal frame of a 3-space. Show that one can describe an octahedron centered at O with vertices $(0, 0, \dots, 0, \pm 1, 0, \dots, 0)$. The vertices are on the axes and the main diagonals have length 2. How would you describe a cross-polytope in an n-dimensional real space ?

1.3.2 Groups

1.3.3 Archimedian polyhedra, prisms and anti-prisms

1.3.4 Kepler, Catalan, Alice Boole Stott, Coxeter

1.3.5 Golden ratio

1.4 Answers to some questions asked by the students

1. When is a polyhedron said to be *regular* ?

   When all its faces are isometric (= equal) and are regular polygons and when there are the same number of faces meeting at each vertex.

2. Why are there not more than 5 convex regular polyhedra ? Why are there exactly 5 ?

   There are at least 5, since we have shown explicitely the 5. There can not be more because the sum of the angles at one vertex has to be less than 360° or 2π radians.

3. Is there some historical story behind polyhedra ?

   Yes, many stories in fact. Usualy, one begins with Plato and ends with symetry groups. In Plato, one finds the following relation between the basic elements and the polyhedra :

tetrahedron	fire
cube	earth
octahedron	air
dodecahedron	ether or universe
icosahedron	water

   It is also the basic asumption of modern physics that the classification of elementary particles has to be done using basic groups. You may find easily many stories on internet.

4. French words

polyhedron	polyèdre
edge	arrête
vertex	sommet
face	face

5. If you inscribe a convex regular polyhedron in a sphere of diameter d, what is the length of its edges ?

   Let us call a the length of one edge and d the length of a diameter of the smallest ball containing the polyhedron. Then

polyhedron	$\frac{a}{d}$
tetrahedron	$\frac{\sqrt{6}}{12}$
cube	$\frac{\sqrt{3}}{12}$
octahedron	$\frac{\sqrt{2}}{8}$
dodecahedron	$\frac{\sqrt{2}(\sqrt{5}-1)}{24}$
icosahedron	$\frac{\sqrt{2}\sqrt{\sqrt{5}-1}}{8\sqrt[4]{5}}$

Many other quantities may be computed.

6. Let P be a convex regular polyhedron, let A be its area and Vol its volume. For which P, is the ration $\frac{A}{Vol}$ biggest ? smallest ?

polyhedron	a$\frac{A}{Vol}$
tetrahedron	$6\sqrt{6} \approx 14,7$
cube	6
octahedron	$3\sqrt{6} \approx 7,35$
dodecahedron	$\frac{3}{2}\sqrt{2\sqrt{5}}(3-\sqrt{5})\sqrt{\sqrt{5}-1} \approx 7,88$
icosahedron	$3\sqrt{3}(3-\sqrt{5}) \approx 3,97$

Wikipedia (on "platonic solids") :

"Among the Platonic solids, either the dodecahedron or the icosahedron may be seen as the best approximation to the sphere. The icosahedron has the largest number of faces, the largest dihedral angle, and it hugs its inscribed sphere the tightest. The dodecahedron, on the other hand, has the smallest angular defect, the largest vertex solid angle, and it fills out its circumscribed sphere the most."

Chapitre 2

Conic sections

If we consider a plane P, let $(O, \vec{\imath}, \vec{\jmath})$ be an orthonormal frame. We denote Ox the line through O in the direction of $\vec{i}$, Oy the line through O in the direction of $\vec{j}$. A point M has coordinates x and y if and only if

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$$

We denote that $M(x, y)$.

If we consider a space E, let $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$ be an orthonormal frame. A point M has coordinates x, y and z if and only if

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

We denote that $M(x, y, z)$. Similar notations for Ox, Oy and Oz as for the plane.

2.1 Line equations and plane equations

2.1.1 Lines

A line D in P parallel to Oy, going through a point $A(x_A, y_A)$ is the set of points $M(x, y)$ such that

$$x = x_A$$

We say that $x = x_A$ is the equation of D. The equation of a line parallel to Ox through the point $B(x_B, y_B)$ is

$$y = y_B$$

The general equation of a line is

$$\alpha x + \beta y + \gamma = 0 \qquad \text{with } \alpha \neq 0 \text{ or } \beta \neq 0.$$

A line secant with Ox in $A(a, 0)$ and with Oy in $B(0, b)$ where $A \neq O$ and $B \neq O$ has the equation

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

2.1.2 Planes

The general equation of a plane is

$$\alpha x + \beta y + \gamma z + \delta = 0 \qquad \text{with } \alpha \neq 0 \text{ or } \beta \neq 0 \text{ or } \gamma \neq 0.$$

It is parallel to the plane xOy if $\beta = \gamma = 0$, it is parallel to Oz if $\gamma = 0$.

A plane secant with Ox in $A(a, 0, 0)$, with Oy in $B(0, b, 0)$ and with $C(0, 0, c)$ where $A \neq O$, $B \neq 0$ and with Oz in $C \neq O$ has the equation

$$\boxed{\tfrac{x}{a} + \tfrac{y}{b} + \tfrac{z}{c} = 1}$$

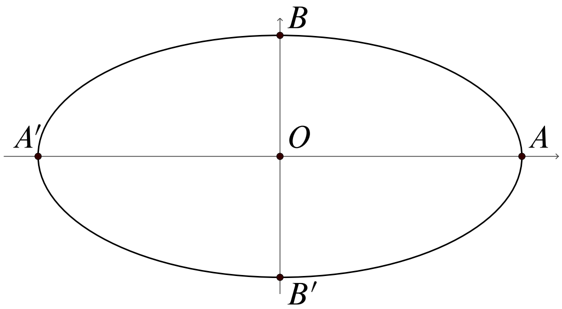
2.2 Ellipses

2.2.1 Cartesian equation

Let $a > b > 0$. The curve described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the following ellipse where $O(0, 0)$, $A(a, 0)$, $B(0, b)$, $A'(-a, 0)$ and $B'(0, -b)$:



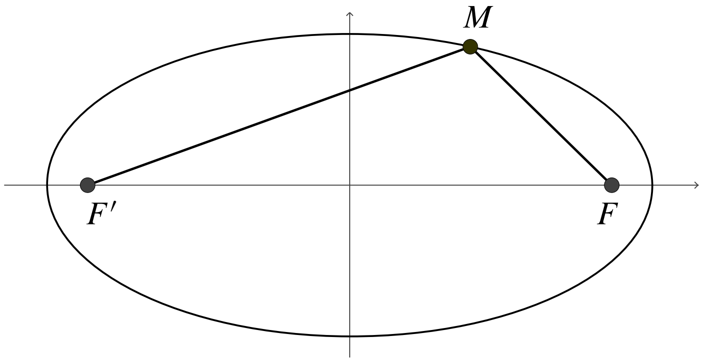
$$OA = OA' = a$$

$$OB = OB' = b$$

The point O is the *center* of the ellipse. The points A and A' are the *vertices* and the points B and B' the *co-vertices* The segment $A'A$ is called the *major axis* and the segment $B'B$ the minor axis if as usual $B'B < A'A$. The number a is half the length of the major axis and b is half the length of the minor axis.

2.2.2 The gardener's ellipse

Plant two poles F' and F in the soil and draw a rope closed by a knot around and stretch it with a third pole M, but this third pole is movable. It will describe an ellipse and inside that ellipse you may plant nice flowers. Let us suppose that the two fixed poles are distant by a distance $2c$ and the closed rope has length $2c + 2a$, then the ellipse will have a as half major axis and $b = \sqrt{a^2 - c^2}$ as half minor axis.

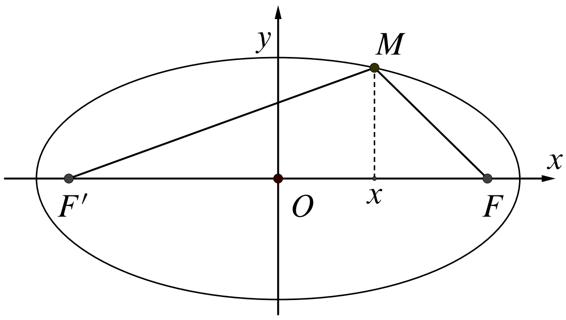


Theorem. Let $a > c > 0$ and let F' and F be two points such that $F'F = 2c$. The set E of points M such that

$$MF + MF' = 2a$$

is the ellipse with center the middle of the segment $F'F$, major axis the line $F'F$, half major axis a and half minor axis $b = \sqrt{a^2 - c^2}$.

Proof. Choose the midpoint O of the segment $F'F$ as origine of the coordinates and the line $F'F$ as x-axis, oriented in such a way that $F(c, 0)$ and $F'(-c, 0)$. Call x and y the coordinates of point M.



By Pythagoras's theorem we have

$$MF' = \sqrt{(x + c)^2 + y^2} \qquad \text{and} \qquad MF = \sqrt{(c - x)^2 + y^2}$$

The point M belongs to E if and only if $MF' + MF = 2a$, or $(MF' + MF)^2 = 4a^2$ that is

$$(x + c)^2 + y^2 + (c - x)^2 + y^2 + 2\sqrt{(x + c)^2 + y^2}\sqrt{(c - x)^2 + y^2} = 4a^2$$

Simplyfying and putting the square root alone on one side we get the equivalent relation

$$\sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2} = 2a^2 - (x^2 + y^2 + c^2)$$

or

$$\begin{cases} (x^2 + y^2 + c^2)^2 - 4c^2x^2 = (2a^2 - (x^2 + y^2 + c^2))^2 \\ x^2 + y^2 \leqslant 2a^2 - c^2 \end{cases}$$

At the end we will see that the inequality is a consequence of the equality. Thus we need not take the inequality into account. We get thus the equivalent relation

$$-4c^2x^2 = 4a^4 - 4a^2(x^2 + y^2 + c^2)$$

or

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

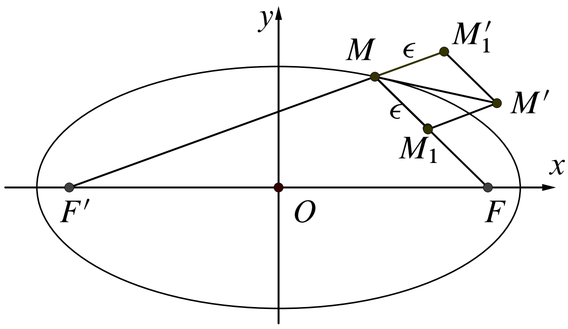
that is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

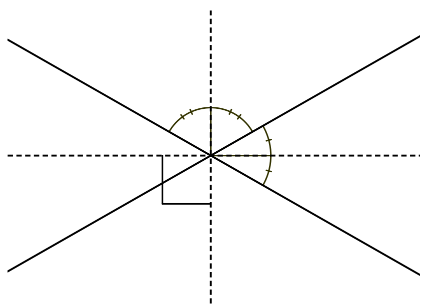
and finally we check the inequation which we may write $x^2 + y^2 \leqslant a^2 + b^2$, which is clear since the ellipse is inside the rectangle with sides parallel to the axis and going through the vertices and co-vertices (notice that $a^2 + b^2$ is the radius of the circumcircle of that rectangle). $\square$

The tangents to the ellipse.

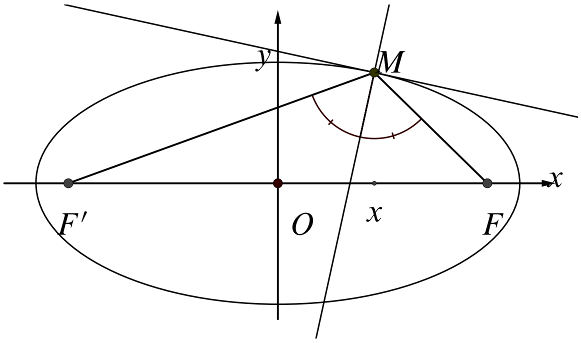
Let us move the point M along the ellipse. We decompose the movement into two virtual movements : first we add a small length ϵ to MF' getting M_1' but to keep $MF' + MF$ constant we must withdraw ϵ from MF getting M_1. Combining these two virtual movements we get a first order approximation of the movement along E, that is a movement along the tangent to E. Thus the tangent is the bisector of the angle $\widehat{M_1MM_1'}$.



Remember that the bisectors of adjacent supplementary angles are orthogonal



Thus the tangent to the ellipse at M is orthogonal to the bisector of angle made by the lines joining M to the foci.



Consequence. If the ellipse is constructed in some reflecting matter and some ray (light or sound or ...) is emitted from F it will be reflected into a ray going through F'.

2.2.3 Parametric equations of the ellipse and affine transformation

When the *parameter* θ describes the interval $[0, 2\pi[$, the point $M(R\cos\theta, R\sin\theta)$ describes the circle C with center O and radius R. We say that the circle C admits the parametric equations

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ \theta \in [0, 2\pi[\end{cases}$$

There are other paramtric equations. For instance the circle C minus the point $(-1, 0)$ admits the following parametric equation

$$\begin{cases} x = R\dfrac{1-t^2}{1+t^2} \\ y = R\dfrac{2t}{1+t^2} \\ t \in \mathbb{R} \end{cases}$$

A point $M(x, y)$ belongs to the ellipse E defined above if and only if $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ or if and only if there is a number θ in $[0, 2\pi[$ such that $\frac{x}{a} = \cos\theta$ and $\frac{y}{b} = \sin\theta$. That means that E admits the following parametric equations

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \\ \theta \in [0, 2\pi[\end{cases}$$

The tangents to the ellipse.

We may use these parametric equations to get easily the slope of the tangent to the ellipse at a point (x, y) where the parameter is θ. Compute the differentials

$$\begin{cases} \mathrm{d}x = -a\sin\theta\mathrm{d}\theta \\ \mathrm{d}y = b\cos\theta\mathrm{d}\theta \end{cases}$$

Then the slope of the tangent at (x, y) is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b}{a}\frac{1}{\tan\theta} = -\frac{b^2}{a^2}\frac{x}{y}$$

(at the points A and A', the tangents to E are parallel to the Oy-axis and thus have no slope or an infinite slope).

Theorem. The tangent to an ellipse E, with foci F' and F, at one of its points M is an exterior bisector of the angle $\widehat{F'MF}$.

The exterior bisector of an angle is the line orthogonal to its bisector.

Proof. To show the theorem we just have to show that the slope of the tangent is the same as the slope of the vector

$$\overrightarrow{V} = \frac{1}{\left\|\overrightarrow{F'M}\right\|}\overrightarrow{F'M} + \frac{1}{\left\|\overrightarrow{MF}\right\|}\overrightarrow{MF}$$

Let us compute the slope of $\overrightarrow{V}$. In our previous frame $(O, \vec{\imath}, \vec{\jmath})$

$$\overrightarrow{F'M}\left|\begin{matrix} x+c \\ y \end{matrix}\right. \qquad \overrightarrow{MF}\left|\begin{matrix} c-x \\ -y \end{matrix}\right.$$

Thus

$$\begin{aligned}\overrightarrow{V} &= \frac{1}{\sqrt{(x+c)^2+y^2}}\Big((x+c)\vec{\imath} + y\vec{\jmath}\Big) + \frac{1}{\sqrt{(c-x)^2+y^2}}\Big((c-x)\vec{\imath} + y\vec{\jmath}\Big) \\ &= \Big(\frac{x+c}{\sqrt{(x+c)^2+y^2}} + \frac{c-x}{\sqrt{(c-x)^2+y^2}}\Big)\vec{\imath} + \Big(\frac{y}{\sqrt{(x+c)^2+y^2}} + \frac{-y}{\sqrt{(c-x)^2+y^2}}\Big)\vec{\imath}\end{aligned}$$

Let us call $m_{\overrightarrow{V}}$ the slope of the vector $\overrightarrow{V} = x_{\overrightarrow{V}}\vec{\imath} + y_{\overrightarrow{V}}\vec{\jmath}$. We have $m_{\overrightarrow{V}} = \frac{y_{\overrightarrow{V}}}{x_{\overrightarrow{V}}}$. Thus after reducing the fractions to the same denominator, we get

$$\begin{aligned}m_{\overrightarrow{V}} &= \frac{y\sqrt{(c-x)^2+y^2} - y\sqrt{(x+c)^2+y^2}}{(x+c)\sqrt{(c-x)^2+y^2} + (c-x)\sqrt{(x+c)^2+y^2}} \\ &= \frac{y\Big(\sqrt{(c-x)^2+y^2} - \sqrt{(x+c)^2+y^2}\Big)\Big(\sqrt{(c-x)^2+y^2} + \sqrt{(x+c)^2+y^2}\Big)}{\Big((x+c)\sqrt{(c-x)^2+y^2} + (c-x)\sqrt{(x+c)^2+y^2}\Big)\Big(\sqrt{(c-x)^2+y^2} + \sqrt{(x+c)^2+y^2}\Big)} \\ &= \frac{y\big((c-x)^2+y^2-(x+c)^2-y^2\big)}{(x+c)\big((c-x)^2+y^2\big) + (c-x)\big((x+c)^2+y^2\big) + 2c\sqrt{x^2+y^2+c^2+2cx}\sqrt{x^2+y^2+c^2-2cx}} \\ &= \frac{-4cxy}{-4cx^2 + 2c\big(x^2+y^2+c^2\big) + 2c\sqrt{\big(x^2+y^2+c^2\big)^2 - 4c^2x^2}} \\ &= \frac{-2xy}{-x^2+y^2+c^2+\sqrt{\big(x^2+y^2+c^2\big)^2 - 4c^2x^2}}\end{aligned}$$

Now we have to simplify the quantity $Z = \sqrt{\left(x^2 + y^2 + c^2\right)^2 - 4c^2x^2}$.

We use $x = a\cos\theta$, $y = b\sin\theta$ and $c^2 = a^2 - b^2$ to get :

$$\begin{aligned} Z^2 &= \left(a^2\cos^2\theta + b^2\sin^2\theta + a^2 - b^2\right)^2 - 4c^2a^2\cos^2\theta \\ &= \left((a^2 - b^2)\cos^2\theta + a^2\right)^2 - 4a^2c^2\cos^2\theta \\ &= \left(c^2\cos^2\theta + a^2\right)^2 - 4a^2c^2\cos^2\theta \\ &= \left(a^2 - c^2\cos^2\theta\right)^2 \end{aligned}$$

Since $Z > 0$ and $a > c$, we have

$$Z = a^2 - c^2\cos^2\theta$$

and thus

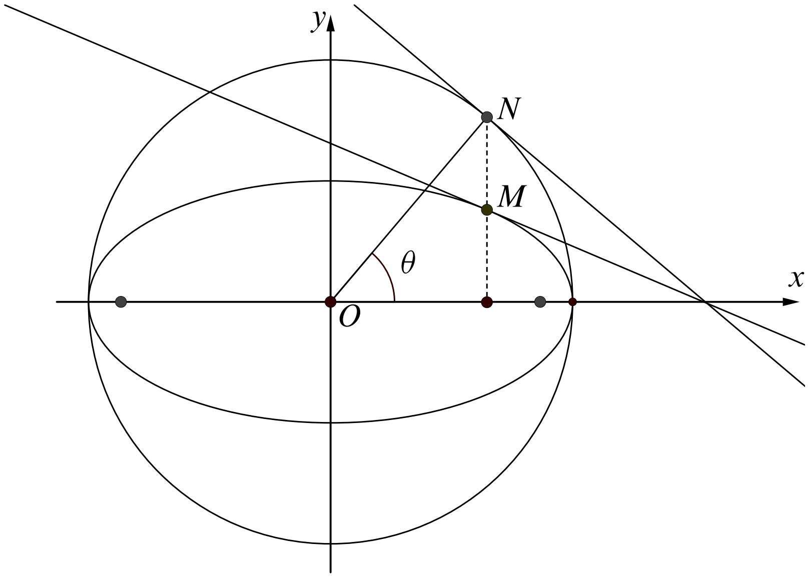
$$\begin{aligned} m_{\overrightarrow{V}} &= \frac{-2xy}{-x^2 + y^2 + c^2 + a^2 - c^2\cos^2\theta} \\ &= \frac{-2ab\cos\theta\sin\theta}{-a^2\cos^2\theta + b^2\sin^2\theta + c^2 + a^2 - c^2\cos^2\theta} \\ &= \frac{-2ab\cos\theta\sin\theta}{(a^2 + b^2 + c^2)\sin^2\theta} = \frac{-2ab\cos\theta}{2a^2\sin\theta} = -\frac{b}{a}\frac{1}{\tan\theta} \end{aligned}$$

We get the same result as by the "intuitive" method of first-order virtual deplacements.

The affine transformation that transforms a circle into an ellipse.

We consider the affine transformation of the plane that transforms each point (x, y) into $(x, \frac{b}{a}y)$. Since it is linear it transforms lines into lines. The points on the Ox-axis are invariant. The circle with center O and radius a which has the equation $x^2 + y^2 = a^2$ is transformed into the curve following the equation $x^2 + \left(\frac{a}{b}y\right)^2 = a^2$, or dividing by a^2

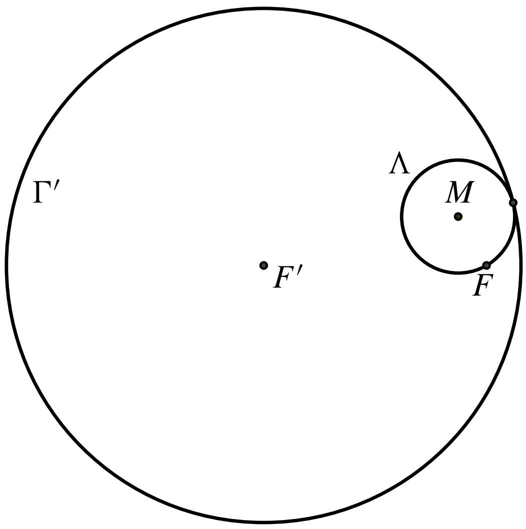
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

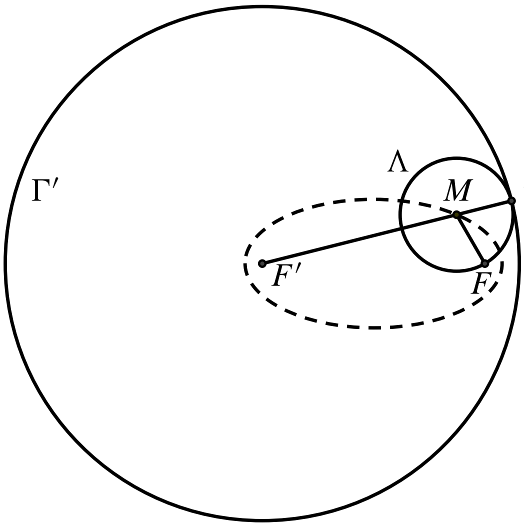


The slope of the line ON, where N has coordinates $(a\cos\theta, a\sin\theta)$ is $\tan\theta$. Thus the slope of the tangent to the circle at N is $-\dfrac{1}{\tan\theta}$. The image of that tangent by the affine transformation transforming the circle into the ellipse is the tangent to the ellipse at the point $M(a\cos\theta, b\sin\theta)$ and it has a slope $\frac{b}{a}\times(-\frac{1}{\tan\theta}) = -\frac{b}{a}\frac{1}{\tan\theta}$.

2.2.4 Two other definitions of an ellipse

Locus of the center of a circle going through a given point and interiorly tangent to a given circle





Theorem. Let Γ' be a circle with center F' and radius $2a$ and let F be a point inside that circle. The set of points M wich are center of circles going through F and tangent to Γ' is the ellipse with foci F' and F and with half major axis a.

Proof. Let M be the center of a circle Λ going through F and tangent to Γ' and call T the point of contact belonging to both circles Γ' and Λ. The point T belongs to the line $F'M$ and M is between F' and T, thus

$$F'M + MT = 2a$$

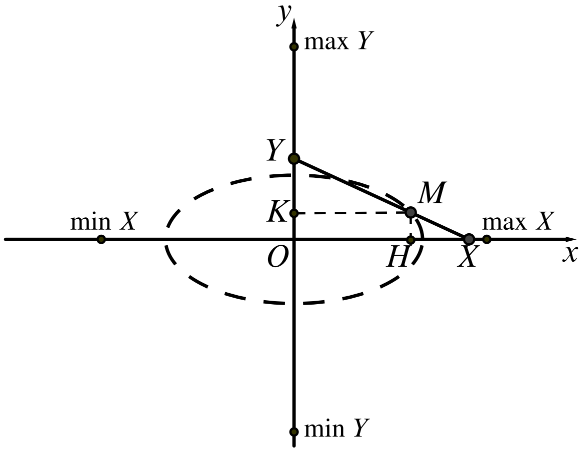
Since T and F belong to a circle with center M, we have $MT = MF$ and thus

$$MF' + MF = 2a$$

Reciprocally, let M belong to the ellipse E, set of points such that $MF' + MF = 2a$. Then MF' is smaller than $2a$ and thus M is interior to the circle Γ'. The line $F'M$ cuts the circle at a point T. We have $F'T = 2a$ or $F'M + MT = 2a$; thus $MF = MT$ and the circle centered at M and tangent to Γ at the point T goes through the point F. $\square$

Locus of a fixed point of a rod whose ends move on the axis Ox and Oy

Theorem. Let $a \geqslant b$ and let XY be a segment of constant length $a + b$, moving in such a way that X belongs to Ox and Y to Oy. Let M be the point on the segment XY such that $MY = a$ and $MX = b$. The locus of the point M is the ellipse with major axis Ox, minor axis Oy, half major axis a and half minor axis b.



Proof. Call $(s, 0)$ the coordinates of X, $(t, 0)$ the coordinates of Y and (x, y) the coordinates of M. Let H be the orthogonal projection of M on Ox and K the orthogonal projection of M on Oy.

Suppose M is on the segment YX and is such that $YM = a$ and $MX = b$. Since the lines MH and YO are parallel

$$\frac{x}{a} = \frac{s}{a+b}$$

Since the lines MK and XO are parallel

$$\frac{y}{b} = \frac{t}{a+b}$$

Thus

$$t = \frac{x}{a}(a+b) \qquad \text{and} \qquad s = \frac{y}{b}(a+b)$$

Finally, since the triangle XOY is rectangle in O, $s^2 + t^2 = (a+b)^2$ and thus

$$\left(\frac{x}{a}\right)^2 (a+b)^2 + \left(\frac{y}{b}\right)^2 (a+b)^2 = (a+b)^2$$

which means that M belongs to the ellipse E with major axis Ox, minor axis Oy, half major axis a and half minor axis b.

Reciprocally, let M be a point of E. The ordinate y of M is less than b or equal to b. Thus the circle with center M and radius b will cut the axis Ox in one or two points. If there are two points call $X(s, 0)$ the point such that $|s| > |x|$. Call $Y(0, t)$ the intersection of the line XM and the Ot-axis. Put $YX = \ell$, we have $MX = a$ and $MY = \ell - b$. Projecting M on the axis, we get

$$\frac{x}{\ell - b} = \frac{s}{\ell} \qquad \text{and} \qquad \frac{y}{b} = \frac{t}{\ell}$$

By Pythagoras' theorem, we have $\ell^2 = s^2 + t^2$, thus

$$1 = \left(\frac{x}{\ell - b}\right)^2 + \left(\frac{y}{b}\right)^2$$

Since M belongs to the ellipse $1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ and thus $\ell - b = a$ or $\ell = a + b$. $\square$

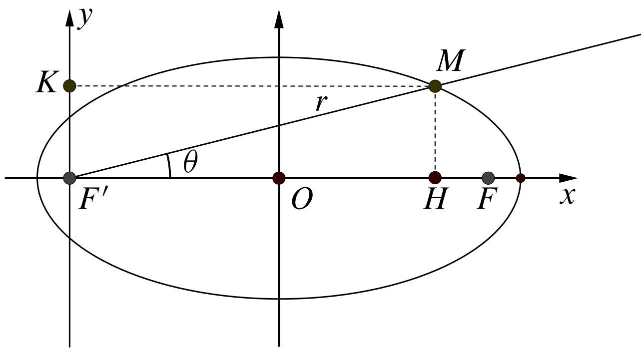
2.2.5 The polar equation with center one focus of an ellipse

Theorem. Let E be an ellipse with one focus F' and major axis $F'x$. The polar equation of E is

$$r = \frac{p}{1 + e\cos\theta}$$

where the number e and the length p are related to the parameters a, b and $c = \sqrt{a^2 - b^2}$ by

$$p = \frac{b^2}{a} \qquad \text{and} \qquad e = \frac{c}{a}$$



Proof. In the frame $(F', \vec{\imath}, \vec{\jmath})$ the equation of the ellipse may be written

$$\frac{(r\cos\theta + c)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1$$

or

$$b^2(r^2\cos^2\theta + 2c\cos\theta r + c^2) + a^2r^2\sin^2\theta = a^2b^2$$

It is a quadratic equation in r (remember that $c^2 - a^2 = -b^2$)

$$(a^2\sin^2\theta + b^2\cos^2\theta)r^2 + 2b^2c\cos\theta r - b^4 = 0$$

The reduced discriminant is[1]

$$\begin{aligned}\Delta' &= (b^2c\cos\theta)^2 - (a^2\sin^2\theta + b^2\cos^2\theta)(-b^4)\\ &= b^4(a^2\cos^2\theta - b^2\cos^2\theta + a^2\sin^2\theta + b^2\cos^2\theta)\\ &= b^4a^2\end{aligned}$$

Thus

$$r = \frac{b^2c\cos\theta \pm b^2a}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{b^2c\cos\theta \pm b^2a}{a^2 - c^2\cos^2\theta}$$

[1] The discriminant of ax^2+bx+c is $\Delta = b^2-4ac$; the reduced discriminant of $ax^2+2b'x+c$ is $\Delta' = b'^2-ac = \frac{1}{4}\Delta$. It is convenient to use Δ' when $b = 2b'$, where b' is simple.

We define e and p by

$$e = \frac{c}{a} \qquad \text{and} \qquad p = \frac{b^2}{a}$$

We have two solutions

$$r = p\frac{e\cos\theta + 1}{1 - e^2\cos^2\theta} \qquad \text{and} \qquad r = p\frac{e\cos\theta - 1}{1 - e^2\cos^2\theta}$$

Since $1 - e^2\cos^2\theta = (1 + e\cos\theta)(1 - e\cos\theta)$, we can simplify and get

$$r = p\frac{1}{1 - e\cos\theta} \qquad \text{and} \qquad r = -p\frac{1}{1 + e\cos\theta}$$

But these two equations give the same curve since the two points with polar coordinates[2] (r, θ) and $(-r, \theta + \pi)$ are in fact the same point. Thus the polar equation of the ellipse is

$$\boxed{r = \frac{p}{1 - e\cos\theta}}$$

Application : Kepler's second and first laws

A massive point (like the earth) in a central field (like the gravitational field of the sun) in $\frac{1}{r}$ is attracted by a force in $\frac{1}{r^2}$ which means that its movement is determined by the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\overrightarrow{F'M} = -k\frac{1}{F'M^3}\overrightarrow{F'M}$$

or

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(r\cos\theta) = -\frac{k}{r^2}\cos\theta \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(r\sin\theta) = -\frac{k}{r^2}\sin\theta \end{cases}$$

As in mechanics, we denote the derivation with respect to time by a point above the quantity derived. The system becomes

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\dot r\cos\theta - r\sin\theta\dot\theta) = -\frac{k}{r^2}\cos\theta \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\dot r\sin\theta + r\cos\theta\dot\theta) = -\frac{k}{r^2}\sin\theta \end{cases}$$

and deriving once again

$$\begin{cases} \ddot r\cos\theta - 2\sin\theta\dot r\dot\theta - r\cos\theta\dot\theta^2 - r\sin\theta\ddot\theta + \dfrac{k}{r^2}\cos\theta = 0 \\ \ddot r\sin\theta + 2\cos\theta\dot r\dot\theta) - r\sin\theta\dot\theta^2 + r\cos\theta\ddot\theta + \frac{k}{r^2}\sin\theta = 0 \end{cases}$$

[2] In fact, when $r > 0$, the couple $(-r, \theta + \pi)$ are not polar coordinates : the polar coordinates of the point described by $(-r, \theta + \pi)$ on a curve are (r, θ).

or

$$\begin{cases} (\ddot{r} - r\dot{\theta}^2 + \dfrac{k}{r^2})\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta = 0 \\ (\ddot{r} - r\dot{\theta}^2 + \frac{k}{r^2})\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta = 0 \end{cases} \quad \begin{vmatrix} \cos\theta \\ \sin\theta \end{vmatrix} \quad \begin{vmatrix} -\sin\theta \\ \cos\theta \end{vmatrix}$$

Multiplying the first equation by $\cos\theta$, the second by $\sin\theta$ and summing we get

$$\ddot{r} - r\dot{\theta}^2 + \frac{k}{r^2} = 0 \qquad (1)$$

Multiplying by $-\sin\theta$ and $\cos\theta$, we get

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$$

which means that $r^2\dot{\theta}$ is constant. This quantity is the *areal speed* of the moving point : it is the speed of growing of the area swept by the radius vector. This is the second of Kepler's laws : the areal speed is constant for a point moving in a central field in $1/r$. Let us call this constant A, we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{A}{r^2} \qquad (2)$$

We want to show that the curve described by the point is a conic section, possibly an ellipse. For that we need to consider r as a function of θ instead of t. To make things easier we introduce $1/r$ as a function u of θ, that is

$$\frac{1}{r} = u(\theta)$$

We then have using (2)

$$u'(\theta) = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{1}{r^2}\dot{r}\frac{1}{\dot{\theta}} = -\frac{1}{A}\dot{r}$$

and again with the help of (2)

$$u''(\theta) = -\frac{1}{A}\ddot{r}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{1}{A}\ddot{r}\frac{1}{\dot{\theta}} = -\frac{1}{A^2}r^2\ddot{r}$$

Multiplying (1) bt $-\frac{1}{A^2}r^2$, we get

$$u''(\theta) - (-\frac{1}{A^2}r^2)r\dot{\theta}^2 + (-\frac{1}{A^2}r^2)\frac{k}{r^2} = 0$$

Using (2) once again, we have

$$u''(\theta) + u(\theta) = \frac{k}{A^2}$$

The general solution of this differential equation is easy to find. It is depending on two constants λ and θ_0 (the choice of the minus sign is arbitrary : we take $-\lambda < 0$ to get the point move in the positive way)

$$u(\theta) = \frac{k}{A^2} - \lambda \cos(\theta - \theta_0)$$

By the choice of the $F'x$-axis, we may suppose that $\theta_0 = 0$. Put $p = \frac{A^2}{k}$ and $e = \lambda p$, we have

$$u(\theta) = \frac{1}{p} - \frac{e}{p} \cos\theta$$

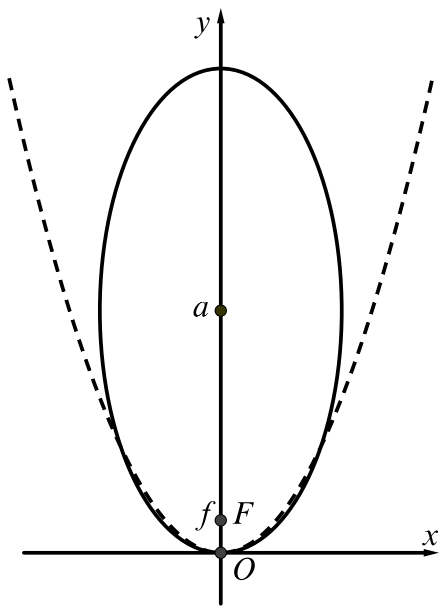
and finally

$$\boxed{r = \frac{p}{1 - e\cos\theta}}$$

Indeed if $e < 1$, the curve is an ellipse. What happens if $e = 0$ or $e > 1$? (The value of e depends on the initial values of the movement).

2.2.6 Cartesian equation of an ellipse in a frame centered at one of its vertices

For the use in next section, let us compute the equation of the ellipse wich has one of its vertices in O and the corresponding focus F on the Oy-axis. Let us use as parmeters the half-major axis a and the ordinate f of F.



The equation of that ellipse is

$$\frac{x^2}{f(2a-f)} + (\frac{y-a}{a})^2 = 1$$

or

$$\frac{ax^2}{2af - f^2} + \frac{y}{a} - 2y = 0$$

or else

$$y = \frac{1}{4f - 2\frac{f^2}{a}} + \frac{1}{2a}y$$

Let $y \longrightarrow +\infty$, we get the usual equation of a parabola

$$y = \frac{1}{4f}x^2$$

This shows that a parabola may be seen as the limit of an ellipse with one focus at infinity.

2.3 Parabola

The curve with equation $y = \alpha x^2$ is a parabola. The same is true for

$$y^2 = 2px$$

Théorème. Let D be a line and F a point that does not belong to D. The locus of the points equidistant from F and D is a parabole. The point F is called the *focus* and D its *directrix*.
Explicitely. The set of points M such that

$$MF = MH$$

where H is the orthogonal projection of M on D.
Proof. Let K be the orthogonal projection of F on D and call O the midpoint of the segment KH. Choose as Ox-axis the line through O orthogonal to D, positively oriented in the direction from K towards F. Choose O as the origin of the frame of the plane. Put

$$p = 2\,OF$$

Thus the coordinates of F are $(\frac{1}{4}p, 0)$ and those of K are $(-\frac{1}{4}p, 0)$. Thus

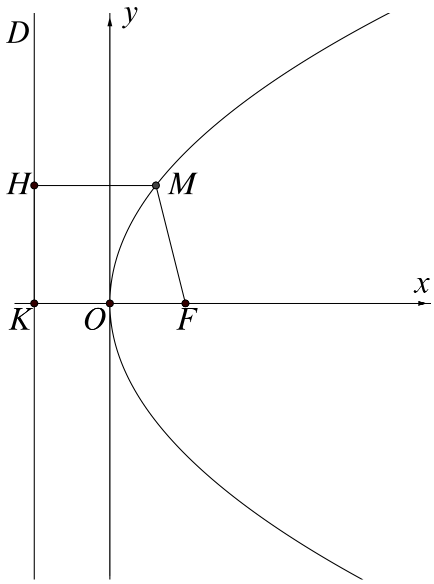
$$MH = x + \frac{1}{2}p \qquad \text{and} \qquad MF^2 = (x - \frac{1}{2}p)^2 + y^2$$

The equation of the locus is then

$$(x - \frac{1}{2}p)^2 + y^2 = (x + \frac{1}{2}p)^2$$

or

$$y^2 = 2px$$



Comment. The locus of the points M such that $MF = eMH$, where $0 < e < 1$, is an ellipse. If $e > 1$, we get a hyperbola.

2.4 Hyperbola

Reread the section about ellipses and change what has to be changed to get the results about the hyperbolae.

2.4.1 Reduced equation

Hyperbola H

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The new aspect is the existence of asymptotes with equations

$$y = \pm \frac{b}{a} x$$

2.4.2 Definition with foci

The hyperbola H is the set of points M such that

$$|MF - MF'| = 2a$$

2.4.3 Parametric equations

$$\begin{cases} x = \epsilon a \cosh t \\ y = b \sinh t \\ t \in \mathbb{R} \end{cases}$$

2.4.4 Other definitions

Let Γ be a circle of center F' and radius $2a$ and let F be a point outside of the disc with border Γ. The Hyperbola H is the locus of the centers M of the circles going through F and tangent to the circle Γ. One branch corresponds to the situation where the two circles are exterior to one an other, the other branch when the circle with center F' is inside the circle with center M.

2.4.5 Polar equation

Théorème. The set of points with polar equation

$$r = \frac{p}{1 - e\cos\theta}$$

with $e > 1$ is a hyperbola.

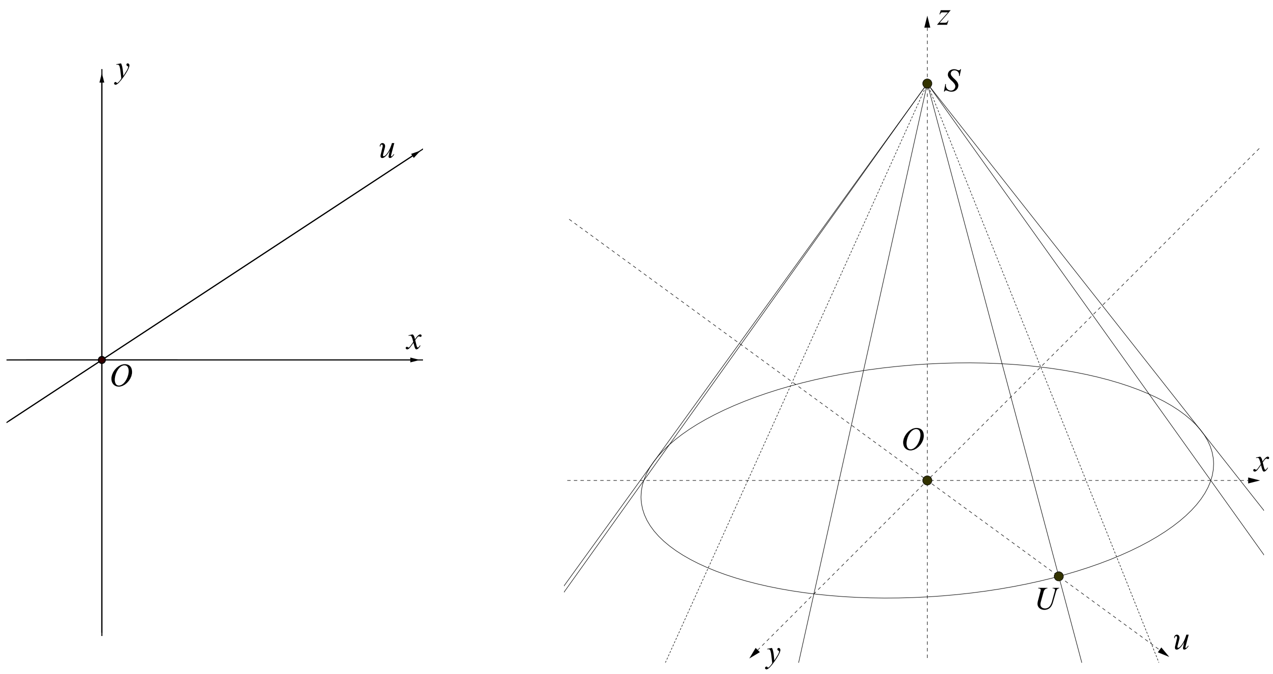
2.5 Sand cones

If you put dry sand in a heap or mound or dune by pouring it from one point it will form a cone with the same slope in all directions.

If you have sand on a solid plane made of metal or wood and if you open a circular hole in that plane you get a cone "uppside down". If your plane is just a halfplane, the sand will take the shape of a plane.

2.5.1 Equation of sand cones

Let us suppose the apex S of our cone has coordinates $(0, 0, z_0)$. Because of the symetry of space the basis of our cone on the plane $z = 0$ will be a circle. Let us call R the radius of that circle. Let Ou be any axis through O in the xOy-plane and call U the point on that circle with positive abscissa in the uOz-plane. The slope of the line SU is a constant characteristic of the sand used. We will denote that slope by m.



The equation of the *generators* of the cone in the plane uOz, that is the lines intersection of that plane with the cone are

$$\frac{u}{R} + \frac{z}{mR} = 1 \qquad \text{and} \qquad \frac{u}{-R} + \frac{z}{mR} = 1$$

or

$$\frac{u}{R} + \frac{z}{mR} - 1 = 0 \qquad \text{and} \qquad -\frac{u}{R} + \frac{z}{mR} - 1 = 0$$

The union of these two lines has the equation

$$(\frac{u}{R} + \frac{z}{mR} - 1)(-\frac{u}{R} + \frac{z}{mR} - 1) = 0$$

or

$$(\frac{z}{mR} - 1)^2 - \frac{u^2}{R^2} = 0$$

Developing and multiplying by $-R^2$ we get

$$u^2 - \frac{1}{m^2}z^2 + \frac{2R}{m}z - R^2 = 0$$

Let us rotate these two lines around the Oz-axis. For every direction we have

$$u^2 = x^2 + y^2$$

Thus the equation of the cone is

$$\boxed{x^2 + y^2 - \frac{1}{m^2}z^2 + \frac{2R}{m}z - R^2 = 0}$$

If we consider a sand cone whose Oz-axis has coordinates (x_0, y_0), we have the equation

$$(x - x_0)^2 + (y - y_0)^2 - \frac{1}{m^2}z^2 + \frac{2R}{m}z - R^2 = 0$$

or

$$\boxed{x^2 + y^2 - \frac{1}{m^2}z^2 - 2x_0x - 2y_0y + \frac{2R}{m}z + x_0^2 + y_0^2 - R^2 = 0}$$

2.5.2 Equation of sand cones upside-down or sand holes

If there is sand on a plane and this sand can fall through a circular hole the border of the sand tkaes the shape of a cone with its apex below the plane suporting the sand. We have choosen the frame in such a way that this plane has equation

$$z = 0$$

The equation is the same as the preceding one in which we cange the apex from (x_0, y_0, mR) into $(x_0, y_0, -mR)$. Thus we get

$$\boxed{x^2 + y^2 - \frac{1}{m^2}z^2 - 2x_0x - 2y_0y - \frac{2R}{m}z + x_0^2 + y_0^2 - R^2 = 0}$$

2.5.3 Intersection of a sand cone with a plane with same slope as the generators of the cone

Let us take the axis of the cone as Oz-axis and the Ox-axis parallel to the plane. The equation of the plane may be written

$$\frac{y}{b} + \frac{z}{mb} = 1$$

where b is a parameter describing the distance between the plane and the origin. We may write this equation as

$$y + \frac{1}{m}z = b$$

If we take the intersection of this plane with the cone

$$x^2 + y^2 - \frac{1}{m^2}z^2 + \frac{2R}{m}z - R^2 = 0$$

we get the curve described by

$$\begin{cases} x^2 + (y + \frac{1}{m}z)(y - \frac{1}{m}z) + \frac{2R}{m}z - R^2 = 0 \\ y + \frac{1}{m}z = b \end{cases}$$

or

$$\begin{cases} x^2 + b(b - \frac{2}{m}z) + \frac{2R}{m}z - R^2 = 0 \\ y + \frac{1}{m}z = b \end{cases}$$

or

$$\begin{cases} x^2 = -\frac{2(R-b)}{m}z + R^2 - b^2 \\ y + \frac{1}{m}z = b \end{cases}$$

which means that the curve in the plane $y + \frac{1}{m}z = b$ is projecting itself on the xOz-plane onto the parabola with equation $x^2 = -\frac{2(R-b)}{m}z + R^2 - b^2$. Thus this intersection is a parabola.

To describe the actual sand cone and to see the intersection, we have to suppose that $0 < b < R$. How do you interpret the minus sign in front of the coefficient of z ? How do you interpret the fact that this coefficient becomes positive when $b > R$?

2.5.4 Intersection of two sand cones upside-down

We suppose the plane is covered with sand and make two circular holes. The sand makes thus two cones which are upside-down but with the same slope m. Let us call R and R' the corresponding parameters and let us suppose $R \geqslant R'$. Let us suppose the axis located at $(0, 0)$ and $(0, y_0)$. The intersection is then the curve with equation

$$\begin{cases} x^2 + y^2 - \frac{1}{m^2}z^2 - \frac{2R}{m}z - R^2 = 0 \\ x^2 + y^2 - \frac{1}{m^2}z^2 - 2y_0 y - \frac{2R'}{m}z + y_0^2 - R^2 = 0 \end{cases}$$

Let us keep the first equation and replace the second one by the difference of the two. We get

$$\begin{cases} x^2 + y^2 - \frac{1}{m^2}z^2 - \frac{2R}{m}z - R^2 = 0 \\ \frac{2(R-R')}{m}z - 2y_0 y + y_0^2 = 0 \end{cases}$$

The intersection of the two cones is thus reduced to the intersection of a cone with a plane. The slope of this plane is

$$\frac{2y_0}{\frac{2(R-R')}{m}} = \frac{my_0}{R-R'}$$

Since the circles in the plane do not cut each other, we have $y_0 > R + R'$; but $R + R' > R - R'$. Thus the slope of the plane is greater than the slope of the cone. The intersection is thus a hyperbola, in fact just an arc of a hyperbola.

2.5.5 Intersection of two sand cones one upright and one upside-down

We suppose the plane is covered with sand poured from one point above the plane having one circular hole. The intersection will be the intersection of two cones, one upright and one upside down. Let us suppose the upright cone has its axis with coordinates $(0, 0)$. The curve have the equation

$$\begin{cases} x^2 + y^2 - \frac{1}{m^2}z^2 + \frac{2R}{m}z - R^2 = 0 \\ x^2 + y^2 - \frac{1}{m^2}z^2 - 2y_0 y - \frac{2R'}{m}z + y_0^2 - R^2 = 0 \end{cases}$$

Let us keep the first equation and replace the second one by the difference of the two. We get

$$\begin{cases} x^2 + y^2 - \frac{1}{m^2}z^2 - \frac{2R}{m}z - R^2 = 0 \\ \frac{2(R+R')}{m}z + 2y_0 y - y_0^2 = 0 \end{cases}$$

The intersection of the two cones is thus reduced to the intersection of a cone with a plane. The absolute value of the slope of this plane is

$$\frac{2y_0}{\frac{2(R+R')}{m}} = \frac{my_0}{R+R'}$$

The two circles have to be one inside the other, thus $y_0 < R$ and $y_0 < R + R'$. The absolute value of the slope of the plane is then less than the slope of the cone ; the intersection is an ellipse. Here we may obtain a complete ellipse if we use enough sand !

Chapitre 3

Graphs

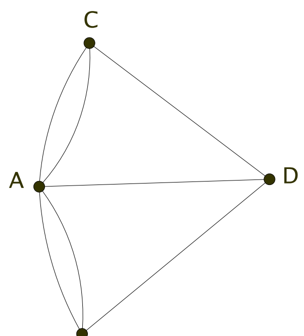
We begin with points that we call vertices. We call the set of these vertices X. We suppose this set finite and call n the number of elements of X, that is the cardinal of X is n.

Then we have edges. Each edge has two ends. If you think of an edge as a path between two vertices, you can think the way as a one-way path or a both-ways path. If the paths are thought as one-way, we have a *directed graph* . We are going to look at undirected graphs, thus both ends of an edge play the same role.

3.1 Königsberg's bridges

The problem submitted to Euler was the following :

In Königsberg there is an island A in a river. Let us call the two banks B and C. After the island the river splits in two rivers forming a new piece of land between them ; we call D this piece of land. There are 2 bridges between A and B, 2 bridges between A and C, 1 between A and D, 1 between B and D and 1 between C and D. Is it possible to travel through the town using each bridge one time and only one time. We may represent the situation by the following graph :



Euler solves the problem and generalizes it to a general graph. He defines the degree of a vertex as the number of edges having this vertex as an endpoint. Under a walk along the graph, each time you go through one vertex, you come with one edge and leave by an other thus without changing the parity of the degree of that vertex. Two exceptions : the departure point and the arrival point. If there were a solution, when you have gone through all bridges, deleting each bridge when you have used it, at the end there would be only vertices with zero edges, that is only vertices with even degrees. The four vertex have odd degrees, thus the problem has no solution.

3.2 Definition

Notation. If X is a set we denote by $X\&X$ the set of subsets of X which have 1 or 2 elements.

$$X\&X = \{\{a_1, a_2\} | a_1 \in X \text{ and } a_2 \in X\}$$

We denote by $\mathcal{P}(X)$ the set of subsets of X, by $\mathcal{P}_2(X)$ the set of subsets of X having 2 elements and by $\mathcal{P}_1(X)$ the set of subsets of X having 1 element. The set $X\&X$ is the union of the sets $\mathcal{P}_2(X)$ and $\mathcal{P}_1(X)$.

$$X\&X = \mathcal{P}_2(X) \cup \mathcal{P}_1(X) = \{A \in \mathcal{P}(X) \mid \text{card } A = 1 \textbf{ or } \text{card } A = 2\}$$

Definitions. A *graph* or *multigraph* G is a triplet $G = (X, E, \Psi)$ where X and E are sets and Ψ is a map from E to $X\&X$. The elements of X are called the *vertices* of the graph G, the elements of E are called the *edges* of the graph G. For each edge e, the elements of $\Psi(e)$ are called the *ends* or *endpoints* of the edge e. If a vertex x is an endpoint of an edge e, we say that this edge e is *adjacent* to x and that the vertex x is *adjacent* to the edge e. An edge e is called a *loop* if $\Psi(e)$ contains only one vertex :"the two endpoints of a loop are the same".
Comment. It is often useful to use oriented edges. In that case one should use $X \times X = X^2$ instead of $X\&X$.

In the following we are concerned only by unoriented graphs where there are no loop and where there is at most one edge for each $\{x_1, x_2\} \in \mathcal{P}_2(X)$. Therefore we will use the following definition where we identify E with $\Psi(E)$.
Definition. A *graph* is a couple (X, E) where $E \subset \mathcal{P}_2(X)$.

3.3 Singleton Graph

K_1
Eulerian and Hamiltoian
Chromatic number : 1
Graph radius = Graph diameter = 0.

3.4 Complete graph of order 2

K_2

Noneulerian and nonhamiltoian

Chromatic number : 2

Graph radius = Graph diameter = 1.

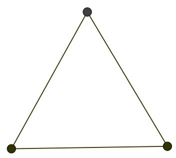
3.5 Complete graph of order 3 : Triangle graph

$K_3 = C_3$

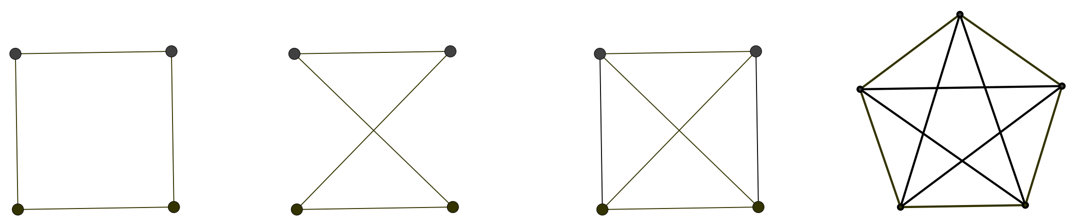
Eulerian and Hamiltoian

Chromatic number : 3

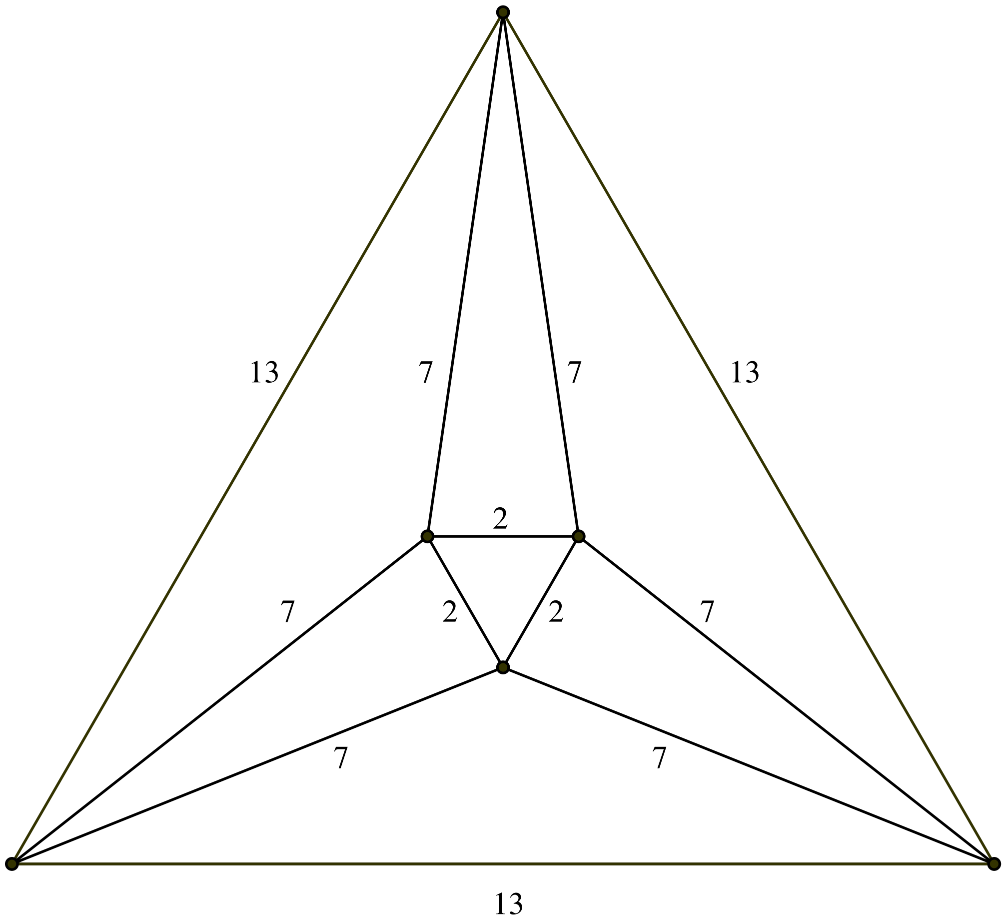
Graph radius = Graph diameter = 1.



3.6 Square graph, K_4 and K_5



3.7 Octahedral graph



Harjoitus 1. Let ABC be an equilateral triangle such that

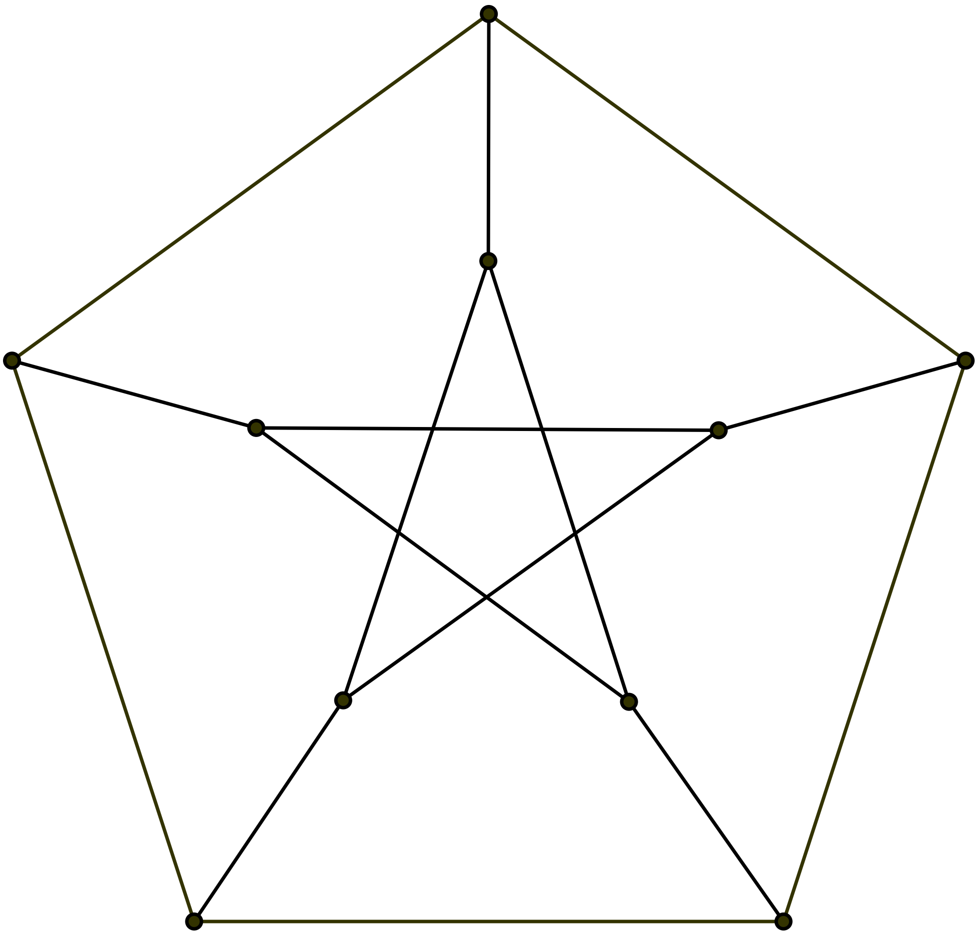
$$BC = CA = AB = 13$$

Let DEF be the triangle inside the triangle ABC such that

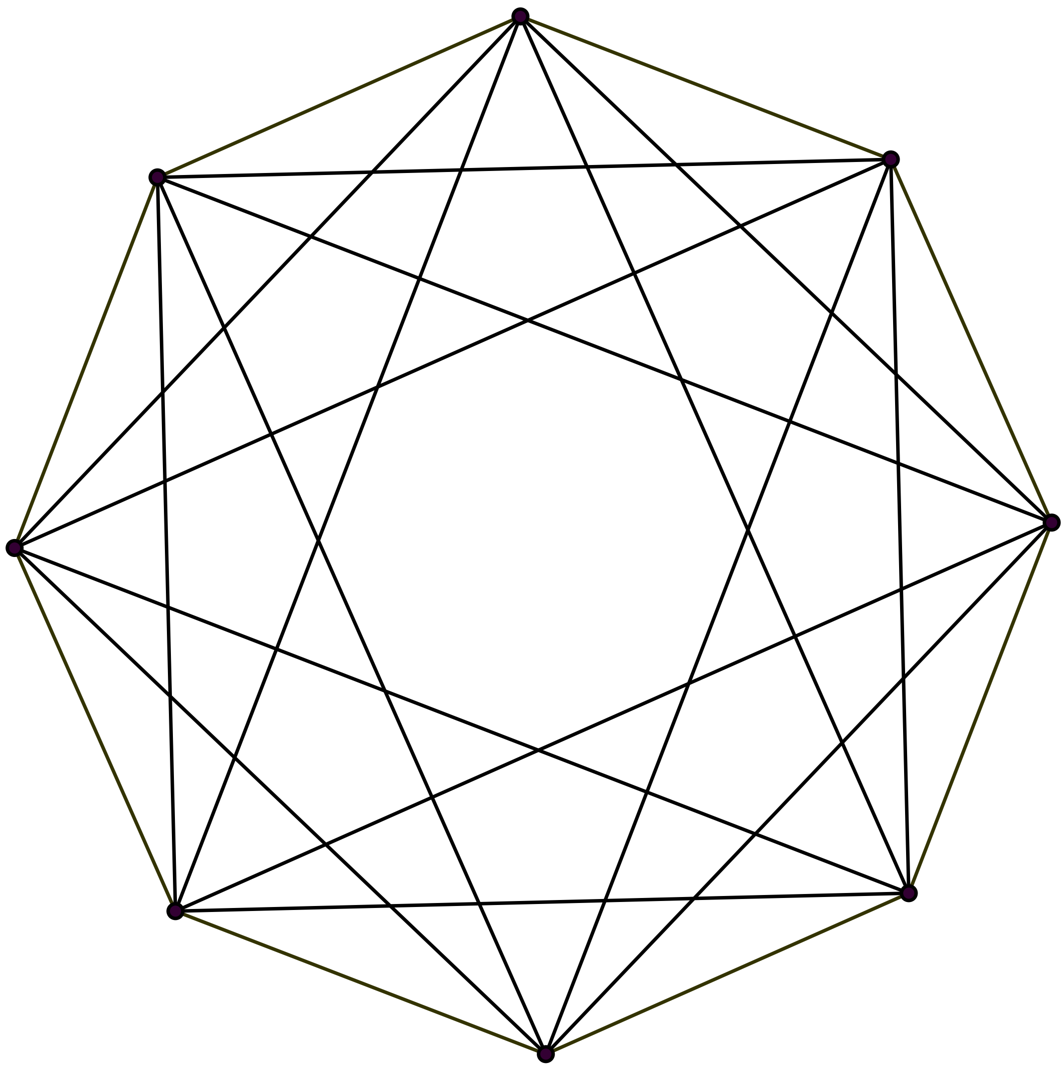
$$BD = CD = CE = AE = AF = BF = 7$$

Then $EF = FD = DE = 2$.

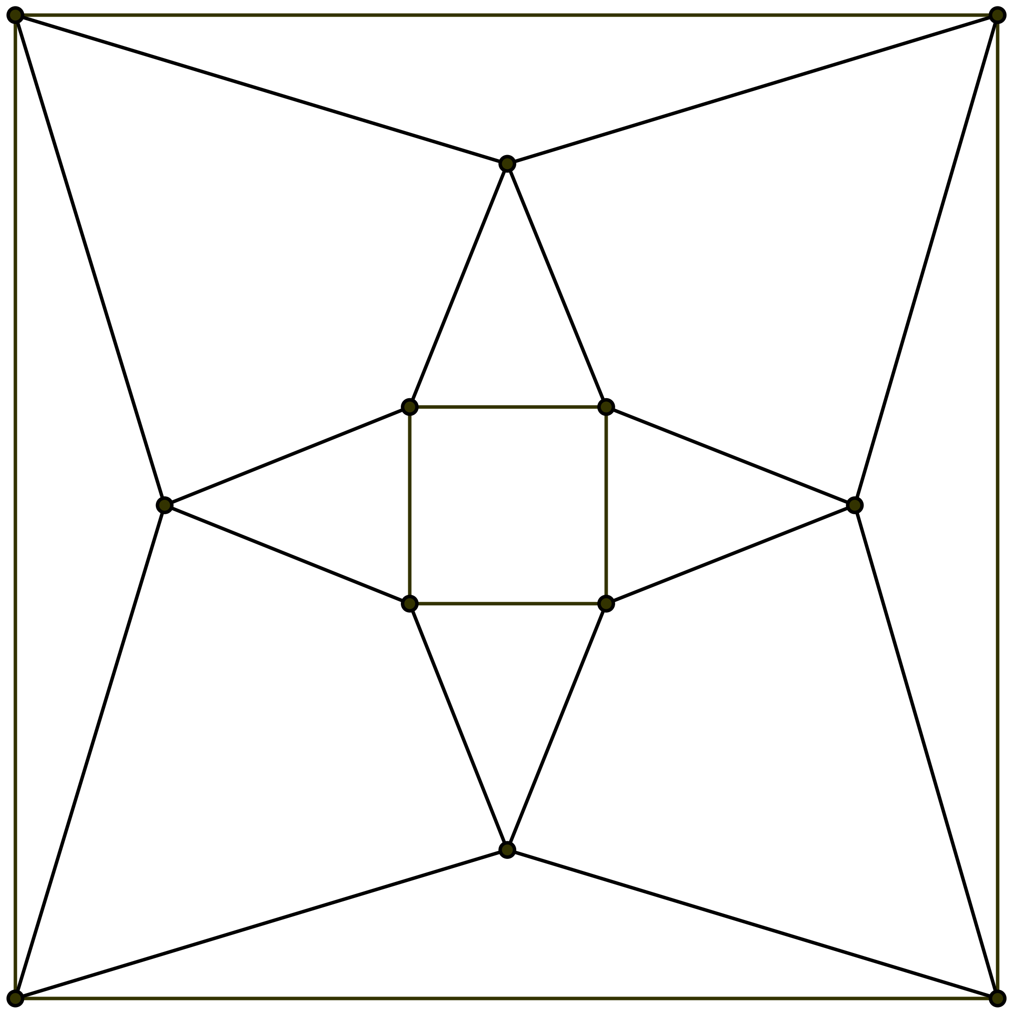
3.8 Petersen graph



3.9 16-cell graph



3.10 Cuboctahedral graph



Chapitre 4

Infinity

4.1 Hilbert's Grand Hotel

The Hilbert's Grand Hotel has infinite many rooms numbered 1, 2, 3, 4 ...

4.1.1 Situation 1

The Hotel is full and a new guest arrives.

Can the manger accommodate the new guest ? - Yes, he can !

There is a simple solution : to ask the guests to change room. Every guest has to move from his room to the room next door. More precisely if n denotes the number of his room, he has to move to the room with number $n+1$. And thus the room number 1 will be free and available for the new guest. We have used the map :

$$f : \mathbb{N}^* \cup \{\text{guest}\} \longrightarrow \mathbb{N}^*, \begin{cases} n \longmapsto n+1 \\ \text{guest} \longmapsto 1 \end{cases}$$

4.1.2 Situation 2

The Hotel is full and each guest has one friend coming. Can the manger accommodate all these new guests ? - Yes, he can !

There is still a simple solution : to ask the guests to change room. Every guest has to move from his room to the room with double number. More precisely if n denotes the number of his room, he has to move to the room with number $2n$. And thus the rooms with odd numbers will be free and available for the new guest. The friend of the person in room n will be accommodated in room $2n-1$ and himself in room $2n$. We have used the map :

$$f : \mathbb{N}^* \cup \mathbb{N}^* \times \{1\} \longrightarrow \mathbb{N}^*, \begin{cases} n \longmapsto 2n \\ (n,1) \longmapsto 2n-1 \end{cases}$$

4.1.3 Situation 3

The Hotel is full and each guest has 9 friends coming.

Can the manger accommodate all these new guests ? - Yes, he can !

There is still a simple solution : to ask the guests to change room. Every guest has to move from his room to the room with the number which is 10 times the number of the room he had previously. More precisely if n denotes the number of his room, he has to move to the room with number $10n$. And thus the rooms with numbers which are not multiple of 10 will be free and available for the new guests. The friends of the person in room n will be accommodated in room $10(n-1)+1$, $10(n-1)+2$, ..., $10(n-1)+9$, and himself in room $10n$. We have used the map :

$$f : \mathbb{N}^* \cup \mathbb{N}^* \times \{1,2,\ldots,9\} \longrightarrow \mathbb{N}^*, \begin{cases} n \longmapsto 10n \\ (n,j) \longmapsto 10(n-1)+j \end{cases}$$

4.1.4 Situation 4

The Hotel is full and each guest has infinite many friends coming.

Can the manger accommodate all these new guests ? - It depends on the kind of infinity. If it is possible to label the friends of the guest in room n by $\mathbb{N}$ for each n, then yes it is possible !

Here is one solution : label the friends of guest n by $(n,1)$, $(n,2)$, $(n,3)$, ... and give the rooms following the new rule : the friend of the person in room n numbered (n,j) will be accommodated in room $\frac{(n+j-1)(n+j)}{2}+n$, and himself in room $\frac{(n-1)n}{2}+n=\frac{n(n+1)}{2}$. We have used the map :

$$f : \mathbb{N}^* \cup \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{N}^*, \begin{cases} n \longmapsto \frac{n(n+1)}{2} \\ (n,j) \longmapsto n+\frac{(n+j-1)(n+j)}{2} \end{cases}$$

We may illustrate this map by the following tables. In the first table we have (n,j) and n identified with $(n,0)$; in the second table the corresponding image by f.

$1=(1,0)$	$2=(2,0)$	$3=(3,0)$	...	$n=(n,0)$	...
$(1,1)$	$(2,1)$	$(3,1)$	...	$(n,1)$	...
$(1,2)$	$(2,2)$	$(3,2)$	...	$(n,2)$	...
$(1,3)$	$(2,3)$	$(3,3)$	...	$(n,3)$	...
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...
$(1,j)$	$(2,j)$	$(3,j)$	...	(n,j)	...
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...

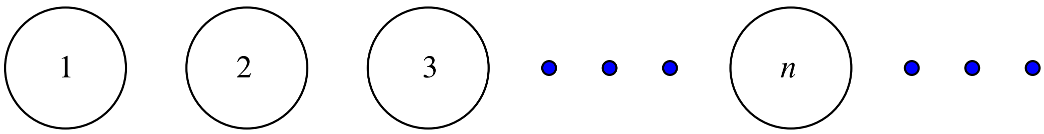
1	3	6	...	$\frac{n(n+1)}{2}$	...
2	5	9	...	$(n,1)$	...
4	8	14	...	$(n,2)$	...
7	13	18	...	$(n,3)$	...
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...
$\frac{j(j+1)}{2}+1$	$\frac{(j+1)(j+2)}{2}+2$	$\frac{(j+2)(j+3)}{2}+3$	...	$\frac{(n+j-1)(n+j)}{2}+n$	...
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...

For instance, the image of $(2, 3)$ is 13.

I have said that the answer depends on the kind of infinity involved. If one guest had as many friends as there are real numbers between 0 and 1, than it would be too many guests for the manager !

4.2 An infinitely deep well

Suppose you have infinite many balls numbered 1, 2, 3, 4, . . .



and a well where you can put the balls, even if there are infinite many of them.

We suppose we can do the operations as fast as we want. Let us suppose we do it each time twice as fast as the previous time : the first operation between 11 o'clock and 11:30, the second between 11:30 and 11:45, the third between 11:45 and 11:52:30, and so on. At noon (or midnight !) that is at 12:00, we'll have done infinite many operations.

4.2.1 First procedure

Rule :

Operation 1. Put the balls 1 to 10 (that is the balls 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10) in the well and extract the ball 10.

Operation 2. Put the balls 11 to 20 (that is the balls 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20) in the well and extract the ball 20.

Operation 3. Put the balls 21 to 30 (that is the balls 21, 22, 23, 24, 25, 26, 27, 28, 29 and 30) in the well and extract the ball 30.

. . ..

Operation n. Put the balls $10(n-1)+1$ to $10n$ in the well and extract the ball $10n$.

And so on . . ..

Question : how many balls are in the well at noon ?

Answer : infinite many. In fact everybody agrees on this answer.

4.2.2 Second procedure

Rule :

Operation 1. Put the balls 1 to 10 (that is the balls 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10) in the well and extract the ball 1.

Operation 2. Put the balls 11 to 20 (that is the balls 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20) in the well and extract the ball 2.

Operation 3. Put the balls 21 to 30 (that is the balls 21, 22, 23, 24, 25, 26, 27, 28, 29 and 30) in the well and extract the ball 3.

. . ..

Operation n. Put the balls $10(n-1)+1$ to $10n$ in the well and extract the ball n.

And so on . . ..

Question : how many balls are in the well at noon ?

Answer : The well is EMPTY. In fact everybody do not agree on this answer. So we have to prove our answer.

Proof. Suppose that at the end the well is not empty. Then the set of the numbers of the balls in the well has a smallest element. Let us call it n_0. But at the operation n_0 that ball has been extracted from the well. Thus we are led to a contradiction. □

4.2.3 Third procedure

Rule :

Operation 1. Put the balls 1 to 10 (that is the balls 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10) in the well and extract one ball at random among the 10 balls in the well.

Operation 2. Put the balls 11 to 20 (that is the balls 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20) in the well and extract one ball at random among the 19 balls still in the well.

Operation 3. Put the balls 21 to 30 (that is the balls 21, 22, 23, 24, 25, 26, 27, 28, 29 and 30) in the well and extract one ball at random among the 28 balls still in the well.

. . ..

Operation n. Put the balls $10(n-1)+1$ to $10n$ in the well and extract one ball at random among the $10n-n+1$ balls still in the well.

And so on . . ..

Question : What is the probability that the well is empty at noon ?

Answer : The probability that well is EMPTY IS EQUAL TO 1. This result can be proven in the theory of probability.

4.3 Decimal development of fractions

4.3.1 A nice proof

Let

$$x = 0,999\,999\,999\,9\ldots$$

multiply both sides by 10

$$10x = 9,999\,999\,999\,9\ldots$$

thus

$$10x = 9 + 0,999\,999\,999\,9\ldots = 9 + x$$

then

$$10x - x = 9$$

or

$$9x = 9$$

and finely

$$x = 1$$

4.3.2 Write the decimal development of fractions with numerator 1

Examples

$$\frac{1}{2} = 0,5 \ ; \ \frac{1}{3} = 0,333\,333\,33 \ ; \ \frac{1}{4} = 0,25 \ ; \ \frac{1}{5} = 0,2 \ ; \ \frac{1}{6} = 0,166\,666\,66$$

$$\frac{1}{7} = 0,142\,857\,142\,857\,142\,857 \ldots \ ; \ \frac{1}{8} = 0,125 \ ; \ \frac{1}{9} = 0,111\,111\,11\ldots \ ; \ \frac{1}{10} = 0,1$$

$$\frac{1}{11} = 0,090\,909\,090\,909\,090\,909 \ldots$$

Describe what happens.

Why is the period of the development of $\frac{1}{n}$ at most $n-1$? can it happen that the period is equal to $n-1$? We call period the minimum number of digits which are repeated at infinity. For instance : $\frac{243}{26} = 9,346\,153\,846\,153\,846\,153\,846\,153\,846 \ldots$ has a period equal to 6 : the sequence 461538 is repeating itself to infinity.

Do you know or can you imagine numbers whose development never becomes periodic ?

When the development is periodic after some decimals, can you find back the fraction ?

Example 1.

Let $a = 0,722\,222\ldots$. Write a as a fraction.

We have $a = 0,7 + \frac{1}{10}0,222\,222 \ldots$. Put $\frac{p}{q} = 0,222\,222 \ldots$, we have $\frac{10p}{q} = 2,222\,222 \ldots$ thus the quotient of the division of $10p$ by q is 2 and the remainder should be p since the period is 1, and we should get on with the same operation at each iteration. Thus $10p = 2q + p$ or $9p = 2q$ and $\frac{p}{q} = \frac{2}{9}$. Finely $a = 0,7 + \frac{1}{10}\frac{2}{9} = \frac{65}{90} = \frac{13}{18}$.

Example 2.

Let $a = 0,142\,857\,142\,857\,142\,857\,142\,857 \ldots$. Write a as a fraction : $a = \dfrac{p}{q}$.

The period is 6, thus

$$\begin{aligned}
10p &= q + r_1\\
10r_1 &= 4q + r_2\\
10r_2 &= 2q + r_3\\
10r_3 &= 8q + r_4\\
10r_4 &= 5q + r_5\\
10r_5 &= 7q + p
\end{aligned}$$

multiply the last relation by 1, The last but one relation by 10, the one before by 100, …, the first one by 100 000. Add and simplify. Then

$$1\,000\,000p = 100\,000q + 40\,000q + 2\,000q + 800q + 50q + 7q + p$$

and

$$999\,999p = 142\,857q$$

thus $a = \frac{142\,857}{999\,999} = \frac{1}{7}$.
Other method : Use the fondamental formula of analysis :

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

and thus

$$\frac{1}{1-\frac{1}{10^p}} = 1 + 10^{-p} + (10^{-p})^2 + (10^{-p})^3 + \cdots + +(10^{-p})^n + \cdots$$

Then

$$\begin{aligned} a &= 142\,857 \times 10^{-6}(1 + 10^{-p} + (10^{-p})^2 + (10^{-p})^3 + \cdots + +(10^{-p})^n + \cdots) \\ &= \frac{142\,857 \times 10^{-6}}{1-10^{-6}} = \frac{142\,857}{10^6-1} = \frac{142\,857}{999\,999} = \frac{1}{7} \end{aligned}$$

Example 3.
Let $a = 9,346\,153\,846\,153\,846\,153\,8\ldots$. Write a as a fraction : $a = \dfrac{p}{q}$.
Answer :

$$\begin{aligned} a &= 9,3 + 0,046\,153\,8(1 + 10^{-6} + 10^{-12} + \cdots) = 9,3 + 0,046\,153\,8\frac{1}{1-10^{-6}} \\ &= \frac{93}{10} + \frac{461\,538}{999\,990} = \frac{93\,461\,445}{9\,999\,990} = \frac{1701}{182} = 9 + \frac{63}{182} = \frac{93}{10} + \frac{3}{65} \end{aligned}$$

Harjoitus 1. Find n and d relatively prime, such that $\frac{n}{d} = 0,315\,757\,575\,757\,575\,757\,575\ldots$.

Harjoitus 2. Find n and d relatively prime, such that $\frac{n}{d} = 0,316\,216\,216\,216\,216\,216\,216\ldots$.

4.4 Developments in bases three and two

4.4.1 In base three

Digits : 0, 1 and 2.
Integers : 0, 1, 2, 1times three, 1times three+1, 1times three+2,
2times three, 2times three+1, 2times three+2,
1times three times three, …

or simpler : 0, 1, 2, 10, 11, 12,
20, 21, 22,
100, ...

Following numbers : 101, 102, 110, 111, 112, 120, 121, ...
Example. 210 211 means in decimal writing $1 + 1 \times 3 + 2 \times 3^2 + 0 \times 3^3 + 1 \times 3^4 + 2 \times 3^5 = 589$.

The other way round : starting with 589, divide it by 3 the remainder is 1, that we keep as last digit, the quotient is 196 that we divide by 3, and so on.

Real positive numbers between 0 and 1
Example. 0,210211 means

$$2\frac{1}{3} + 1\frac{1}{3^2} + 0\frac{1}{3^3} + 2\frac{1}{3^4} + 1\frac{1}{3^5} + 1\frac{1}{3^6} = \frac{2}{3} + \frac{1}{9} + \frac{2}{81} + \frac{1}{243} + \frac{1}{729}$$

As in the base ten, we have in base three : $0,120\,122\,222\,222\ldots = 0,120\,200\,000\,000\ldots$. The fractions with denominator 3^k have two developments.

4.4.2 In base two

Digits : 0, and 1.
Integers : 0, 1, 1× two, 1× two +1, 1× two × two, 1× two × two+1,
1× two × two+1× two,1× two × two+1× two+1, 1× two × two × two,
1× two × two × two+1, ...

or simpler : 0, 1, 10, 11, 100, 101,
110, 111, 1 000,
1 001, ...

Following numbers : 1 010, 1 011, 1 100, 1 101, 1 110, 1 111, ...
Example. 110 011 means in decimal writing $1 + 1 \times 2 + 0 \times 2^2 + 0 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 = 51$.

The other way round : starting with 51, divide it by 2 the remainder is 1, that we keep as last digit, the quotient is 25 that we divide by 2, and so on.

Real positive numbers between 0 and 1
Example. 0,110011 means

$$1\frac{1}{2} + 1\frac{1}{2^2} + 0\frac{1}{2^3} + 0\frac{1}{2^4} + 1\frac{1}{2^5} + 1\frac{1}{2^6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{32} + \frac{1}{64}$$

As in the base ten, we have in base two : $0,110\,011\,111\,111\ldots = 0,110\,100\,000\,000\ldots$. The fractions with denominator 2^k have two developments.

And now begins my story about Cantor and his marvellous set....