Infinity

§1. Hilbert's Grand Hotel

The Hilbert's Grand Hotel has infinite many rooms numbered 1, 2, 3, 4 …

1.1. Situation 1. The Hotel is full and a new guest arrives. Can the manger accommodate the new guest ?
- Yes, he can. There is a simple solution: to ask the guests to change room. Every guest has to move from his room to the room next door. More precisely if n denotes the number of his room, he has to move to the room with number $n+1$. And thus the room number 1 will be free and available for the new guest. We have used the map:

$$f:\mathbb{N}\longrightarrow\mathbb{N},\ n\longmapsto n+1$$

1.2. Situation 2. The Hotel is full and each guest has one friend coming. Can the manger accommodate all these new guests ?
- Yes, he can. There is still a simple solution: to ask the guests to change room. Every guest has to move from his room to the room with double number. More precisely if n denotes the number of his room, he has to move to the room with number $2n$. And thus the rooms with odd numbers will be free and available for the new guest. The friend of the person in room n will be accommodated in room $2n-1$ and himself in room $2n$. We have used the map:

$$f:\mathbb{N}\longrightarrow\mathbb{N},\ n\longmapsto 2n$$

1.3. Situation 3. The Hotel is full and each guest has 9 friends coming. Can the manger accommodate all these new guests ?
- Yes, he can. There is still a simple solution: to ask the guests to change room. Every guest has to move from his room to the room with the number which is 10 times the number of the room he had previously. More precisely if n denotes the number of his room, he has to move to the room with number $10n$. And thus the rooms with numbers which are not multiple of 10 will be free and available for the new guests. The friends of the person in room n will be accommodated in room $10(n-1)+1$, $10(n-1)+2$, … , $10(n-1)+9$, and himself in room $10n$. We have used the map:

$$f:\mathbb{N}\longrightarrow\mathbb{N},\ n\longmapsto 10n$$

1.3. Situation 3. The Hotel is full and each guest has infinite many friends coming. Can the manger accommodate all these new guests ?
- It depends on the kind of infinity. If it is possible to label the friends of the guest in room n by n for each n, then yes it is possible! Here is one solution: label the friends

of guest n by $(n,1)$, $(n,2)$, $(n,3)$, … and give the rooms following the new rule: the friend of the person in room n numbered (n,j) will be accommodated in room $(1/2)(n+j-1)(n+j)+j+1$, and himself in room $(1/2)(n+j-1)(n+j)+1$. We have used the map:

$$f: \mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N},\ n \longmapsto (1/2)(n+j-1)(n+j)+j+1$$

I have said that the answer depends on the kind of infinity involved. If one guest had as many friends as there are real numbers between 0 and 1, than it would be too many guests for the manager.

§2. An infinitely deep well

Suppose you have infinite many balls numbered 1, 2, 3, 4, …

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and a dwell where you can put the balls.

We suppose we can do the operations as fast as we want. Let us suppose we do it each time twice as fast as the previous time: the first operation between 11 o'clock and 11:30, the second between 11:30 and 11:45, the third between 11:45 and 11:52:30, and so on. At noon, that is 12:00, we'll have done infinite many operations.

2.1. First procedure.

Rule:

Operation 1. Put the balls 1 to 10 (that is the balls 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10) in the well and extract the ball 10.

Operation 2. Put the balls 11 to 20 (that is the balls 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20) in the well and extract the ball 20.

Operation 3. Put the balls 21 to 30 (that is the balls 21, 22, 23, 24, 25, 26, 27, 28, 29 and 30) in the well and extract the ball 30.

…….

Operation n. Put the balls $10(n-1)+1$ to $10n$ in the well and extract the ball $10n$.

And so on ………..

Question: how many balls are in the well at noon?

2.2. Second procedure.

Rule:

Operation 1. Put the balls 1 to 10 (that is the balls 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10) in the well and extract the ball 1.

Operation 2. Put the balls 11 to 20 (that is the balls 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20) in the well and extract the ball 2.

Operation 3. Put the balls 21 to 30 (that is the balls 21, 22, 23, 24, 25, 26, 27, 28, 29 and 30) in the well and extract the ball 3.

…….

Operation n. Put the balls $10(n\text{-}1)+1$ to $10n$ in the well and extract the ball n.

And so on ………..

Question: how many balls are in the well at noon?

2.2. Third procedure.

Rule:

Operation 1. Put the balls 1 to 10 in the well and extract a ball at random.

Operation 2. Put the balls 11 to 20 in the well and extract a ball at random.

Operation 3. Put the balls 21 to 30 in the well and extract a ball at random.

…….

Operation n. Put the balls $10(n\text{-}1)+1$ to $10n$ in the well and extract a ball at random.

And so on ………..

Question: What is the probability that the well is empty at noon?

§ 3. Decimal development of fractions

3.1. A nice proof

Let

$$x = 0{,}99999...$$

multiply both sides by 10

$$10x = 9{,}99999...$$

thus

$$10x = 9 + 0{,}99999... = 9 + x$$

then

$$10x - x = 9 \quad \text{or} \quad 9x = 9$$

and finely

$$x = 1$$

3.2. Write the decimal development of fractions with numerator 1.

$\frac{1}{2} = 0{,}5$; $\frac{1}{3} = 0{,}333333...$; $\frac{1}{4} = 0{,}25$; $\frac{1}{5} = 0{,}2$; $\frac{1}{6} = 0{,}16666...$; $\frac{1}{7} = 0{,}142857142857142857...$
$\frac{1}{8} = 0{,}125$; $\frac{1}{9} = 0{,}11111...$; $\frac{1}{10} = 0{,}1$; $\frac{1}{11} = 0{,}09090909...$ $\ldots$

Describe what happens.

Why is the period of the development of $\frac{1}{n}$ at most $n-1$? can it happen (that the period is equal to $n-1$)? We call period the minimum number of digits which are repeated at infinity. For instance: $\frac{243}{26} = 9{,}3461538461538461538461538461538...$ has a period equal to 6: the sequence 461538 is repeating itself to infinity.

Do you know or can you imagine numbers whose development never becomes periodic ?

When the development is periodic after some decimals, can you find back the fraction ?

Example 1. $a = 0{,}72222... = 0{,}7 + \frac{1}{10}0{,}22222....$ Put $\frac{p}{q} = 0{,}22222....$, we have $\frac{10p}{q} = 2{,}2222....$ thus the quotient of the division of $10\,p$ by q is 2 and the remainder should be p since the period is 1, and we should get on with the same operation at each iteration. Thus $10p = 2q + p$ or $9p = 2q$ and $\frac{p}{q} = \frac{2}{9}$. Finely $a = 0{,}7 + \frac{1}{10}\frac{2}{9} = \frac{7}{10} + \frac{2}{90} = \frac{65}{90} = \frac{13}{18}$.

Example 2. $a = 0{,}142857142857142857...$ The period is 6, thus

$$\begin{aligned} 10p &= q + r_1 \\ 10r_1 &= 4q + r_2 \\ 10r_2 &= 2q + r_3 \\ 10r_3 &= 8q + r_4 \\ 10r_4 &= 5q + r_5 \\ 10r_5 &= 7q + p \end{aligned}$$

multiply the last but one relation by 10, the one before by 100, …, the first one by 100 000. Then

$$1000000p = 100000q + 40000q + 2000q + 800q + 50q + 7q + p$$

and

$$999999p = 142857q$$

thus $a = \frac{p}{q} = \frac{142857}{999999} = \frac{1}{7}$.

Other method : use the fondamental formula of analysis

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... + x^n + ...$$

and thus

$$\frac{1}{1-\frac{1}{10^p}} = 1 + 10^{-p} + (10^{-p})^2 + (10^{-p})^3 + (10^{-p})^4 + ... + (10^{-p})^n + ...$$

For example :

$$9{,}34615384615 38... = 9{,}3 + 0{,}0461538(1 + 10^{-6} + 10^{-12} + ...) + 9{,}3 + 0{,}0461538 \times \frac{1}{1-10^{-6}}$$

Exercice 1. Find n and d relatively prime, such that $\frac{n}{d} = 0{,}31575757...$

Exercice 2. Find n and d relatively prime, such that $\frac{n}{d} = 0{,}3162162162...$

§ 4. Development in bases 3 and 2

4.1. In base 3

Integers: 0, 1, 2, 1times three, 1times three+1, 1times three+2, 2times three, 2times three+1, 2times three+2, 1times three times three, …

or simpler. 0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, 110, 111, 112, 120, …

Example: 210211 means in decimal writing $1+1\times 3+2\times 3^2+0\times 3^3+1\times 3^4+2\times 3^5=589$.

and the other way round : starting with 589, divide it by 3 the remainder is 1, that we keep as last digit, the quotient is 196 that we divide by 3, and so on.

Real positive numbers between 0 and 1 :

Example. 0,210211 means $2\frac{1}{3}+1\frac{1}{3^2}+0\frac{1}{3^3}+2\frac{1}{3^4}+1\frac{1}{3^5}+1\frac{1}{3^6}=\frac{2}{3}+\frac{1}{9}+\frac{2}{81}+\frac{1}{243}+\frac{1}{729}$.

As in the base 10, we have 0,120122222222…=0,120200000000…. The fractions with denominator 3^k have two developments.

And now begins my story about Cantor and his marvelous set….