Analysis 4

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Abstract

These notes form a body of the course Analysis 4 (Analyysi 4) teached in the University of Joensuu, spring 2002. Since the majority of the textbooks on this level are written in English, it is justified to write these notes in English as well.

The aim of Analysis 4 is to form a "gentle" introduction to linear functional analysis. To form a foundation for the subject, we first recall some basic concepts of linear algebra and metric spaces in Section 1. Many of the most important spaces which arise in functional analysis are spaces of integrable functions. To avoid various drawbacks of the elementary Riemann integral (see Analysis 2), it is necessary to use more flexible tool known as *Lebesgue integration*. Therefore, Section 2 is an introduction to Lebesgue integrals and to measure theory. It is not possible to consider everything what is known in Lebesgue integrals and measure theory in Analysis 4. Therefore the interested reader is invited to proceed to Analysis 5, which classically has contained more careful presentation of these topics.

The actual context of Analysis 4 is in further sections, Sections $3 - n$, $n \geq 3$, where we consider normed spaces (Banach spaces), inner product spaces (Hilbert spaces) and linear operators in the broadness to which our time frame allows us.

These notes form a Version 1.0. The first attempt is always the first attempt and therefore the existence of serious errors in these notes is more than probable.

1 Preliminaries

To a certain extent, functional analysis can be described as infinite-dimensional linear algebra combined with analysis, in order to make sense of ideas such as convergence and continuity. It follows that we will make extensive use of these topics, so in this section we recall various results and notations which are fundamental to the study of functional analysis.

Linear algebra

The following standard sets will be used:

$$\begin{aligned}\mathbb{N} &= \text{the set of positive integers (zero excluded)},\\ \mathbb{R} &= \text{the set of real numbers},\\ \mathbb{C} &= \text{the set of complex numbers}.\end{aligned}$$

The sets $\mathbb{R}$ and $\mathbb{C}$ are algebraic *fields* (kuntia). For convenience, we often let $\mathbb{F}$ to denote either set.

For any $k \in \mathbb{N}$ we let $\mathbb{F}^k = \mathbb{F} \times \cdots \times \mathbb{F}$ denote the Cartesian product (karteesinen tulo) of k copies of $\mathbb{F}$.

Let $f : X \longrightarrow Y$ denote a function or a mapping from X to Y. If $A \subset X$ and $B \subset Y$, we denote

$$f(A) = \{f(x) \mid x \in A\} \quad \text{and} \quad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Definition 1.1 A *vector space* (vektoriavaruus) over $\mathbb{F}$ is a non-empty set V together with addition (a function from $V \times V$ to V) and multiplication (a function from $\mathbb{F} \times V$ to V) such that, for any $\alpha, \beta \in \mathbb{F}$ and any $x, y, z \in V$,

(a) $x + y = y + x, \quad x + (y + z) = (x + y) + z,$

(b) there exists a unique zero element $0 \in V$ such that $x + 0 = x$,

(c) there exists a unique inverse element $-x \in V$ such that $x + (-x) = 0$,

(d) $1 \cdot x = x, \quad \alpha(\beta x) = (\alpha\beta)x,$

(e) $\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x.$

If $\mathbb{F} = \mathbb{R}$ (resp. if $\mathbb{F} = \mathbb{C}$) then V is a real (resp. complex) vector space. Elements of $\mathbb{F}$ are called *scalars*, while elements of V are called *vectors*. The operation $x+y$ is called *vector addition*, while the operation αx is called *scalar multiplication*.

Definition 1.2 Let V be a vector space. A non-empty set $U \subset V$ is a *linear subspace* of V if U is itself a vector space with the same vector addition and scalar multiplication as in V.

Example. $\mathbb{R}^2$ is a linear subspace of $\mathbb{R}^3$.

Theorem 1.3 (Subspace test) *Let V be a vector space (over $\mathbb{F}$) and let $U \subset V$ be a non-empty set. Then U is a linear subspace of V if and only if $\alpha x + \beta y \in U$ for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in U$.*

Example. The right half-plane $H = \{(x, y) \mid x > 0, y \in \mathbb{R}\}$ is not a linear subspace of $\mathbb{R}^2$.

Definition 1.4 Let V be a vector space and let $\mathbf{v} = \{v_1, \ldots, v_k\} \subset V$, $k \geq 1$, be a finite set.

(a) A *linear combination* of the elements of $\mathbf{v}$ is any vector of the form

$$x = \alpha_1 v_1 + \cdots + \alpha_k v_k \in V, \tag{1.1}$$

for any set of scalars $\alpha_1, \ldots, \alpha_k$.

(b) $\mathbf{v}$ is *linearly independent* if the following implication holds:

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \Longrightarrow \quad \alpha_1 = \ldots = \alpha_k = 0.$$

If $\mathbf{v}$ is not linearly independent, $\mathbf{v}$ is *linearly dependent*.

(c) If any $x \in V$ can be represented in the form (1.1) with unique set of scalars $\alpha_1, \ldots, \alpha_k$, then $\mathbf{v}$ is called a *basis* (kanta) for V. These scalars are then called the *components* of x with respect to the basis $\mathbf{v}$.

Example. The set $\mathbb{R}^k$ is a vector space over $\mathbb{R}$ and the set of vectors

$$\hat{e}_1 = (1, 0, \ldots, 0),\ \hat{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots\ ,\ \hat{e}_k = (0, 0, \ldots, 1)$$

is a basis for $\mathbb{R}^k$ often known as the *standard basis* (luonnollinen kanta) for $\mathbb{R}^k$.

Example. The set $\mathbb{C}^k$ is a vector space over $\mathbb{C}$. What might be the standard basis for $\mathbb{C}^k$?

If a given vector space V has a finite basis (including k elements), then V is said to be *finite-dimensional* (*k-dimensional* to be exact), and we write $\dim V = k$. It is possible that $\dim V = \infty$.

Example. Let V be a set of all infinite sequences $x = (x_1, x_2, \ldots)$, $x_n \in \mathbb{C}$, satisfying $\sum_{n=1}^{\infty} |x_n| < \infty$. V endowed with addition $x+y = (x_1+y_1, x_2+y_2, \ldots)$ and scalar multiplication $\alpha x = (\alpha x_1, \alpha x_2, \ldots)$, $\alpha \in \mathbb{F}$, is an infinite dimensional vector space over $\mathbb{F}$. Such a space V is often denoted by ℓ^1.

Definition 1.5 Let S be a set and let V be a vector space over $\mathbb{F}$. We denote the set of functions $f : S \longrightarrow V$ by $F(S, V)$. For any $\alpha \in V$ and any $f, g \in F(S, V)$, we define functions $f + g$ and αf in $F(S, V)$ by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x)$$

for all $x \in S$ (using the vector space operations in V).

Remarks. (1) In the above definition, $F(S, V)$ is a vector space over $\mathbb{F}$. The zero element in $F(S, V)$ is the function which is identically equal to the zero element of V (this is the only non-trivial thing here).

(2) Many of the vector spaces used in functional analysis are of the above form. From now on, whenever functions are added or multiplied by a scalar the process will be as in Definition 1.5.

(3) If S contains infinitely many elements and $V \neq \{0\}$, then $F(S, V)$ is infinite dimensional.

Example. If S is the set of integers $\{1, \ldots, k\}$, then the set $F(S, \mathbb{F})$ can be identified with the space $\mathbb{F}^k$.

Definition 1.6 Let V, W be vector spaces over the same scalar field $\mathbb{F}$. A function $T : V \longrightarrow W$ is called a *linear transformation* (or *mapping*) if, for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in V$,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

The set of all linear transformations $T : V \longrightarrow W$ will be denoted by $L(V, W)$. When $V = W$, we abbreviate $L(V, V)$ to $L(V)$.

Remarks. (1) With the scalar multiplication and vector addition given in Definition 1.5 the set $L(V, W)$ is a vector space — a subspace of $F(V, W)$.

(2) A particularly simple linear transformation in $L(V)$ is the *identity transformation* (identtinen kuvaus) $I_V(x) = x$. If it is clear in what space the transformation is acting on, we simply write $I = I_V$.

(3) From now on, whenever we discuss about linear transformations $T : V \longrightarrow W$, it will be taken for granted that V and W are vector spaces over the same scalar field.

Since linear transformations are functions they can be composed (yhdistää). The following two lemmas are rather elementary but important.

Lemma 1.7 *Let V, W, X be vector spaces and $T \in L(V, W)$, $S \in L(W, X)$. Then $S \circ T \in L(V, X)$.*

Lemma 1.8 *Let V be a vector space, $R, S, T \in L(V)$ and $\alpha \in \mathbb{F}$ (the scalar field). Then:*

(a) $R \circ (S \circ T) = (R \circ S) \circ T$,

(b) $R \circ (S + T) = R \circ S + R \circ T$,

(c) $(S + T) \circ R = S \circ R + T \circ R$,

(d) $I \circ T = T \circ I = T$,

(e) $(\alpha S) \circ T = \alpha(S \circ T) = S \circ (\alpha T)$.

The following lemma gives further properties of linear transformations.

Lemma 1.9 *Let V, W be vector spaces and let $T \in L(V, W)$. Then:*

(a) $T(0) = 0$.

(b) If U is a linear subspace of V then the set $T(U)$ is a linear subspace of W and $\dim T(U) \leq \dim W$.

(c) If U is a linear subspace of W then the set $X = \{x \in V \mid T(x) \in U\}$ is a linear subspace of V.

Linear transformations between finite-dimensional vector spaces are closely related to matrices (matriisit). To this end, for any integers $m, n \geq 1$, let $M_{mn}(\mathbb{F})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{F}$. A typical element of $M_{mn}(\mathbb{F})$ will be written as $[a_{ij}]$ (or $[a_{ij}]_{mn}$ if it is necessary to emphasize the size of the matrix).

Any matrix $C = [c_{ij}] \in M_{mn}(\mathbb{F})$ induces a linear transformation $T_C \in L(\mathbb{F}^n, \mathbb{F}^m)$ as follows. For any $x \in \mathbb{F}^n$, let $T_C(x) = y$, where each element y_i of $y \in F^m$ is defined by

$$y_i = \sum_{j=1}^{n} c_{ij} x_j, \quad 1 \leq i \leq m.$$

In matrix representation,

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

C is called the matrix for the linear transformation T_C.

Metric spaces

Metric spaces are an abstract setting in which to discuss basic analytical concepts such as convergence of sequences and continuity of functions.

Definition 1.10 A *metric* on a set M is a function $d : M \times M \longrightarrow \mathbb{R}$ with the following properties:

(a) $d(x, y) \geq 0$,

(b) $d(x, y) = 0 \iff x = y$,

(c) $d(x, y) = d(y, x)$,

(d) $d(x, z) \leq d(x, y) + d(y, z)$ (the *triangle inequality*).

If d is a metric on M, then the pair (M, d) is called a *metric space.*

Example. For any $k \in \mathbb{N}$, the function $d : \mathbb{F}^k \times \mathbb{F}^k \longrightarrow \mathbb{R}$ given by

$$d(x, y) = \left(\sum_{j=1}^{k} |x_j - y_j|^2 \right)^{1/2}$$

is a metric on $\mathbb{F}^k$. This metric will be called the *standard metric* on $\mathbb{F}^k$ and, unless otherwise stated, $\mathbb{F}^k$ will be regarded as a metric space with this metric.

Any given set M can have more than one metric. This fact is illustrated in the following for $M = \mathbb{F}^k$.

Example. A pair $(\mathbb{F}^k, d_1)$ with $d_1 : \mathbb{F}^k \times \mathbb{F}^k \longrightarrow \mathbb{R}$ defined by

$$d_1(x, y) = \sum_{j=1}^{k} |x_j - y_j|$$

is a metric space.

Definition 1.11 Let (M, d) be a metric space and let $N \subset M$. Define $d_N : N \times N \longrightarrow \mathbb{R}$ by $d_N(x, y) = d(x, y)$ for all $x, y \in N$ (that is, d_N is a restriction of d to the subset N). Then d_N is called the metric *iduced* on N by d (metriikan d indusoima).

Whenever we consider subsets of metric spaces we will regard them as metric spaces with the induced metric, unless otherwise stated.

A *sequence* in a set X is often defined to be a (discrete) function $s : \mathbb{N} \longrightarrow X$. Alternatively, a sequence in X can be regarded as an ordered list of elements of X written in the form $\{x_n\}$ (or $\{x_n\}_{n=1}^{\infty}$) with $x_n = s(n)$ for each $n \in \mathbb{N}$. Compare the notation to the set $\{x_n \mid n \in \mathbb{N}\}$, which has no ordering!

Example. Using the definition of a sequence as a function from $\mathbb{N}$ to $\mathbb{F}$, we see that the space $F(\mathbb{N}, \mathbb{F})$ can be identified with the space consisting of all sequences in $\mathbb{F}$.

A fundamental concept in analysis is the convergence of sequences. Convergence of sequences in metric spaces will be recalled next.

Definition 1.12 A sequence $\{x_n\}$ in a metric space (M, d) converges to $x \in M$ if, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x, x_n) < \varepsilon \quad \text{for all} \quad n \geq N_\varepsilon.$$

The sequence $\{x_n\}$ is called a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon \quad \text{for all} \quad m, n \geq N_\varepsilon.$$

Theorem 1.13 *Suppose that $\{x_n\}$ is a convergent sequence in a metric space (M, d). Then:*

(a) the limit $\lim\limits_{n\to\infty} x_n$ is unique,

(b) any subsequence of $\{x_n\}$ also converges to x,

(c) $\{x_n\}$ is a Cauchy sequence.

Definition 1.14 Let (M, d) be a metric space. For any point $x \in M$ and any number $r > 0$, the set

$$B(x, r) = \{y \in M \mid d(x, y) < r\} \qquad (= B_d(x, r))$$

is said to be the *open ball* with centre x and radius r. The set

$$\overline{B(x, r)} = \{y \in M \mid d(x, y) \leq r\} \qquad (= \overline{B_d(x, r)})$$

is said to be the *closed ball* with centre x and radius r.

Definition 1.15 Let (M, d) be a metric space and let $A \subset M$.

(a) A is *open* if, for each $x \in A$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A$.

(b) A is *closed* if the set $M \setminus A$ is open.

(c) The *closure* of A is denoted by $\bar{A}$.

(d) A is *dense* in M if $\bar{A} = M$.

(e) A is *nowhere dense* in M if $\bar{A}$ has empty interior (interior points in $\bar{A}$ are centers of some open balls in $\bar{A}$).

In real analysis the idea of a continuous function can be defined in terms of the standard metric on $\mathbb{R}$, so the idea can also be extended to the general metric space setting.

Definition 1.16 Let (M, d_M) and (N, d_N) be metric spaces and $f : M \longrightarrow N$ be a function.

(a) f is *continuous at a point* $x \in M$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for $y \in M$,

$$d_M(x, y) < \delta \quad \Longrightarrow \quad d_N(f(x), f(y)) < \varepsilon.$$

(b) f is *continuous* on M if it is continuous at each point of M.

(c) f is *uniformly continuous* (*tasaisesti jatkuva*) on M if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in M$,

$$d_M(x,y) < \delta \implies d_N(f(x), f(y)) < \varepsilon.$$

Remark. For a uniformly continuous function f on M, the number δ can be chosen independently of $x, y \in M$.

Theorem 1.17 *Suppose that (M, d_M) and (N, d_N) are metric spaces and that $f : M \longrightarrow N$. Then:*

(a) f is continuous at $x \in M$ if and only if, for every sequence $\{x_n\} \subset M$ with $x_n \longrightarrow x$, the sequence $\{f(x_n)\} \subset N$ satisfies $f(x_n) \longrightarrow f(x)$,

(b) f is continuous on M if and only if either of the following conditions holds:

(i) for any open set $A \subset N$, the set $f^{-1}(A) \subset M$ is open,

(ii) for any closed set $A \subset N$, the set $f^{-1}(A) \subset M$ is closed.

Definition 1.18 A metric space (M, d) is *complete* (täydellinen) if every Cauchy sequence in (M, d) is convergent. A set $A \subset M$ is complete in (M, d) if every Cauchy sequence lying in A converges to an element of A.

Theorem 1.19 *For each $k \in \mathbb{N}$, the space $\mathbb{F}^k$ with the standard metric is complete.*

Theorem 1.20 (Baire's theorem, version 1) *If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space (M, d), then there exists at least one point in M which is not in any of the sets A_n.*

For the proof of Theorem 1.20, see e. g. *Simmons:* Introduction to Topology and Modern Analysis.

Theorem 1.21 (Baire's theorem, version 2) *If (M, d) is a complete metric space and $M = \cup_{n=1}^{\infty} A_n$, where each $A_n \subset M$ is closed, then at least one of the sets A_n contains an open ball.*

Proof. Suppose on the contrary that none of the sets A_n contains an open ball. Since A_n's are closed (that is, $A_n = \bar{A}_n$), this means that $\{A_n\}$ is a sequence of nowhere dense sets in M (see Definition 1.15) and, by Theorem 1.20, there exists $x \in M$ such that $x \notin \cup_{n=1}^{\infty} A_n$. Hence $M \neq \cup_{n=1}^{\infty} A_n$, a contradiction. □

Remark. A subset of a metric space is called a set of the *first category* if it can be represented as the union of a sequence of nowhere dense sets, and a set of the *second category* if it is not a set of the first category. Baire's theorem — often known as Baire's category theorem — can now be expressed as follows: *Any complete metric space is a set of the second category.*

Definition 1.22 Let M be a topological space (not necessarily a metric space) and Γ be a family of subsets of M. Then Γ is a *covering* of a set $A \subset M$ if and only if $A \subset \cup_{K \subset \Gamma} K$. If Γ contains finitely many (resp. countably many) subsets of M, then Γ is called a *finite cover* (resp. *countable cover*) of A. If Γ contains only open subsets of M, then Γ is an open cover of A. If $\Gamma' \subset \Gamma$ and $A \subset \cup_{K' \subset \Gamma'} K'$, then Γ' is a *subcover* of Γ for A.

In what follows, we are mainly interested in metric spaces.

Definition 1.23 Let M be a topological/metric space. A set $A \subset M$ is *compact* if every open cover of A has a finite subcover. A is *relatively compact* if the closure $\bar{A}$ is compact. If the set M itself is compact, then we say that (M, d) is a *compact topological/metric space.*

The following four theorems are classical.

Theorem 1.24 (Bolzano-Weierstrass theorem) *Every infinite subset of a compact space has at least one cluster point (kasaantumispiste).*

Theorem 1.25 (Lindelöf covering theorem) *Every open covering of a set $A \subset \mathbb{R}^n$ has a countable subcovering.*

Theorem 1.26 (Cantor intersection theorem) *Let $\{Q_j\}$ be a sequence of non-empty closed sets in $\mathbb{R}^n$ such that $Q_1 \supset Q_2 \supset \dots$ Suppose that Q_1 is bounded. Then $S = \cap_{j=1}^{\infty} Q_j$ is closed, bounded and non-empty.*

Theorem 1.27 (Heine-Borel theorem) *Every closed and bounded set $A \subset \mathbb{R}^n$ is compact.*

From the elementary analysis it is known that a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and attains (saavuttaa) its maximum and minimum on the interval $[a, b]$. Our next result generalizes this fact.

Theorem 1.28 *Suppose that (M, d) is a compact metric space and $f : M \longrightarrow \mathbb{R}$ is continuous. Then there exists a constant $b > 0$ such that $|f(x)| \leq b$ for all $x \in M$ (that is, f is bounded). The numbers $\sup\{f(x) \mid x \in M\}$ and $\inf\{f(x) \mid x \in M\}$ exist and are finite. Furthermore, there exist points $x_s, x_i \in M$ such that $f(x_s) = \sup\{f(x) \mid x \in M\}$ and $f(x_i) = \inf\{f(x) \mid x \in M\}$.*

Remark. In the above theorem, we could consider functions $f : M \longrightarrow \mathbb{C}$. These functions would still be finite (by modulus) and the supremum of $|f(x)|$ would exist as a finite number. The infimum of $|f(x)|$, however, would be meaningless in this case.

Definition 1.29 Let (M, d) be a compact metric space. The set of continuous functions $f : M \longrightarrow \mathbb{F}$ will be denoted by $C_F(M)$ or simply by $C(M)$.

Lemma 1.30 *Let (M, d) be a compact metric space. A function $d : C(M) \times C(M) \longrightarrow \mathbb{R}$ defined by*

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in M\}$$

is a metric on $C(M)$.

The metric $d(f, g)$ in the above lemma will be called *uniform metric* and, unless otherwise stated, $C(M)$ will always be assumed to have this metric.

Definition 1.31 Suppose that (M, d) is a compact metric space and $\{f_n\}$ is a sequence in $C(M)$. Let $f : M \longrightarrow \mathbb{F}$ be a function.

(a) $\{f_n\}$ converges *pointwise* to f if $|f_n(x) - f(x)| \longrightarrow 0$ for all $x \in M$.

(b) $\{f_n\}$ converges *uniformly* to f if $\sup\{|f_n(x) - f(x)| \mid x \in M\} \longrightarrow 0$.

Example. Uniform convergence implies pointwise convergence.

Theorem 1.32 *The metric space $C(M)$ is complete*

For the proof of Theorem 1.32, see *Burkill & Burkill:* A Second Course in Mathematical Analysis, pages 50 – 52.