4 Normed spaces

When the vector spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ are pictured in the usual way, we have the idea of the length of a given vector. This is clearly a bonus which gives us a deeper understanding of these vector spaces. When we turn to other (possibly infinite-dimensional) vector spaces, we might hope to get more insight into these spaces if there is some way of assigning something similar to the length of a given vector. This consideration leads to a set of axioms that will define the "norm" of a given vector.

In this section we shall consider some elementary properties of several known normed vector spaces.

Definition of a norm and examples

Until so far, we have treated several normed spaces without referring to norms. Hence, we first define what is meant by a norm and then show that most of the vector spaces we have treated are in fact normed spaces.

Definition 4.1 Let X be a vector space over $\mathbb{F}$. A norm on X is a function $||\cdot|| : X \longrightarrow \mathbb{R}$ such that for all $x, y \in X$ and all $\alpha \in \mathbb{F}$,

(a) $||x|| \geq 0$,

(b) $||x|| = 0$ if and only if $x = 0$ (the zero element in X),

(c) $||\alpha x|| = |\alpha|\,||x||$,

(d) $||x + y|| \leq ||x|| + ||y||$.

A normed space on which there is a norm is called a normed vector space or simply just a normed space. A unit vector in a normed space X with norm $||\cdot||$ is a vector $x \in X$ such that $||x|| = 1$.

As a motivation, we note that the length of a given vector in one of the spaces $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$ satisfies the axioms of a norm. This is a simple consequence of the following:

Example. The function $||\cdot|| : \mathbb{F}^n \longrightarrow \mathbb{R}$ defined by

$$||x|| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}, \quad x = (x_1, \ldots, x_n),$$

is a norm on $\mathbb{F}^n$ called the *standard norm* on $\mathbb{F}^n$.

The solution to the example above follows from

Example. Let X be a finite-dimensional vector space over $\mathbb{F}$ and let $\{e_1, \ldots, e_n\}$ be a linearly independent basis of X. Any $x \in X$ can be written as $x = \sum_{j=1}^{n} \lambda_j e_j$ for unique $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. Then the function $||\cdot|| : X \longrightarrow \mathbb{R}$ defined by

$$||x|| = \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^{1/2}$$

is a norm on X.

Many interesting normed spaces are not finite-dimensional.

Example. Let M be a compact metric space (with any metric). Then the function $||\cdot|| : C_{\mathbb{F}}(M) \longrightarrow \mathbb{F}$ defined by

$$||f|| = \sup\{|f(x)| \mid x \in M\}$$

is a norm on $C_F(M)$ called the *standard norm* on $C_F(M)$.

Example. If $1 \leq p < \infty$, then

$$||f||_p = \left(\int |f|^p \, dm\right)^{1/p}$$

is a norm on L^p called the *standard norm* on L^p. Further,

$$||f||_\infty = \text{ess sup}\,|f(x)|$$

is a norm on L^∞ called the *standard norm* on L^∞.

Example. If $1 \leq p < \infty$, then

$$||\{x_n\}||_p = \left(\sum_n |x_n|^p\right)^{1/p}$$

is a norm on ℓ^p called the *standard norm* on ℓ^p. Further,

$$||\{x_n\}||_\infty = \sup_n |x_n|$$

is a norm on ℓ^∞ called the *standard norm* on ℓ^∞.

In what follows, whenever we consider any of the normed spaces above without mentioning a norm, it will be assumed that the norm on the space is the standard norm.

Example. Let X and Y be vector spaces over $\mathbb{F}$ and let $Z = X \times Y$. Then Z is a vector space over $\mathbb{F}$. Moreover, if $||\cdot||_1$ is a norm on X and $||\cdot||_2$ is a norm on Y then $||(x,y)|| = ||x||_1 + ||y||_2$ is a norm on Z.

Since the norm of a vector is a generalization of the length of a vector in $\mathbb{R}^3$, it is perhaps not surprising that each normed space is a metric space in a very natural way.

Lemma 4.2 *Let X be a vector space with norm $||\cdot||$. If $d : X \times X \longrightarrow \mathbb{R}$ is given by $d(x,y) = ||x-y||$, then d is a metric, that is, (X,d) is a metric space.*

Whenever we use a metric or a metric space concept in a normed space, we will always use the metric assosiated with the norm even if this is not explicitly stated.

Theorem 4.3 *Let X be a vector space over $\mathbb{F}$ with norm $||\cdot||$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to $x, y \in X$, respectively, and let $\{\alpha_n\}$ be a sequence in $\mathbb{F}$ converging to $\alpha \in \mathbb{F}$. Then:*

(a) $|\,||x|| - ||y||\,| \leq ||x-y||$, *(b)* $\lim_{n\to\infty} ||x_n|| = ||x||$,

(c) $\lim_{n\to\infty} (x_n + y_n) = x + y$, *(d)* $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

Finite-dimensional normed spaces

The simplest vector spaces to study are the finite-dimensional ones. We have already seen in an example that each finite-dimensional space has a norm, but this norm depends on the choice of basis. This suggests that there can be many different norms on each finite-dimensional space. Even in $\mathbb{R}^2$ we have already seen that there are at least two norms:

(a) the standard norm $||(x_1, x_2)|| = \sqrt{x_1^2 + x_2^2}$,

(b) the norm $||(x_1, x_2)|| = |x_1| + |x_2|$.

However, if we have two norms on a vector space, it is possible that the metric space properties of the space could be the same for both norms. This happens (see Theorem 4.7), when the norms are equivalent in the sense of the following definition.

Definition 4.4 Let X be a vector space and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on X. The norm $||\cdot||_2$ is equivalent to the norm $||\cdot||_1$ if there exist $M, m > 0$ such that

$$m||x||_1 \leq ||x||_2 \leq M||x||_1$$

holds for all $x \in X$.

Example. Prove that the norms $||\cdot||_a$ and $||\cdot||_b$ above are equivalent in $\mathbb{R}^2$.

We next give two lemmas on a vector space with at least two norms.

Lemma 4.5 *Let X be a vector space and let $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_3$ be three norms on X. Let $||\cdot||_2$ be equivalent to $||\cdot||_1$ and let $||\cdot||_3$ be equivalent to $||\cdot||_2$.*

(a) $||\cdot||_1$ is equivalent to $||\cdot||_2$,

(b) $||\cdot||_3$ is equivalent to $||\cdot||_1$.

Lemma 4.6 *Let X be a vector space and let $||\cdot||_1$ and $||\cdot||_2$ be norms on X. Let d_1 and d_2 be metrics defined by $d_1(x, y) = ||x-y||_1$ and $d_2(x, y) = ||x-y||_2$. Suppose that there exists $K > 0$ such that $||x||_1 \leq K||x||_2$ for all $x \in X$. Let $\{x_n\} \in X$ be a sequence.*

(a) If $\{x_n\}$ converges to x in the metric space (X, d_2), then $\{x_n\}$ converges to x in the metric space (X, d_1).

(b) If $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_2), then $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_1).

The metric space properties of a vector space are the same for equivalent norms, as is seen in the following.

Theorem 4.7 *Let X be a vector space and let $||\cdot||_1$ and $||\cdot||_2$ be equivalent norms on X. Let d_1 and d_2 be metrics defined by $d_1(x, y) = ||x - y||_1$ and $d_2(x, y) = ||x - y||_2$. Let $\{x_n\} \in X$ be a sequence.*

(a) $\{x_n\}$ converges to x in the metric space (X, d_1) if and only if $\{x_n\}$ converges to x in the metric space (X, d_2).

(b) $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_1) *if and only if* $\{x_n\}$ *is a Cauchy sequence in the metric space* (X, d_2).

(c) (X, d_1) *is complete if and only if* (X, d_2) *is complete.*

As far as many metric space properties are concerned, Theorem 4.7 implies that it does not matter which one of the equivalent norms we consider. This is important as sometimes one of the norms is easier to work with than the other.

We next show that any norm on a finite-dimensional vector space X is equivalent to the norm based on the basis of the space and given in an example above.

Theorem 4.8 *Let* X *be a finite-dimensional vector space with norm* $||\cdot||_1$ *and let* $\{e_1, \ldots, e_n\}$ *be a linearly independent basis for* X. *Let* $||\cdot||_2$ *be the norm*

$$||x||_2 = \left(\sum_{j=1}^{n} |\lambda_j|^2 \right)^{1/2}, \tag{4.1}$$

where $x = \sum_{j=1}^{n} \lambda_j e_j \in X$ *(see an example above). Then the norms* $||\cdot||_1$ *and* $||\cdot||_2$ *are equivalent.*

Corollary 4.9 *If* $||\cdot||_1$ *and* $||\cdot||_2$ *are any two norms on a finite-dimensional vector space* X *then they are equivalent.*

Lemma 4.10 *Let* X *be a finite-dimensional vector space and let* $\{e_1, \ldots, e_n\}$ *be a linearly independent basis for* X. *If* $||\cdot||_2 : X \longrightarrow \mathbb{R}$ *is the norm on* X *defined by (4.1) then* X *is a complete metric space.*

Finally, we prove that any finite-dimensional normed space is complete.

Theorem 4.11 *If* $||\cdot||$ *is any norm on a finite-dimensional vector space* X *then* X *is a complete metric space.*

Banach spaces

As in the case of metric spaces, the most important normed spaces are the complete ones. These spaces have a special name: *Banach spaces.*

Definition 4.12 A Banach space is a normed vector space which is complete under the metric associated with the norm.

Theorem 4.13 *The following normed spaces are Banach spaces:*

(a) $C_{\mathbb{F}}(X)$ *space, where* X *is a compact metric space,*

(b) L^p *spaces, where* $1 \leq p \leq \infty$,

(c) ℓ^p *spaces, where* $1 \leq p \leq \infty$,

(d) all complex function spaces mentioned in Section 3,

(e) all finite-dimensional normed vector spaces.

There are, of course, many other Banach spaces than those mentioned in Theorem 4.13.

We shall close this section with an analogue of the absolute convergence test for series, which is valid for Banach spaces.

Definition 4.14 Let X be a normed space and let $\{x_n\} \subset X$ be a sequence. For each $k \in \mathbb{N}$ let $s_k = \sum_{n=1}^{k} x_k$. The series $\sum_{n=1}^{\infty} x_n$ is said to converge if $\lim_{k\to\infty} s_k$ exists in X and, if so, we define

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} s_k.$$

Theorem 4.15 *Let X be a Banach space with norm $||\cdot||$ and let $\{x_n\} \subset X$ be a sequence. If the series $\sum_{n=1}^{\infty} ||x_n||$ converges then the series $\sum_{n=1}^{\infty} x_n$ converges.*

Note that $\mathbb{R}$ is a Banach space with the absolute value being the norm. Therefore, Theorem 4.15 is a generalization of the analogous result for series of real numbers given in Analysis 1.