6 On linear operators

Now that we have studied some of the properties of normed (and inner product) spaces, we turn to look at functions which map one space into another. Since the spaces that we have studied are linear, it is natural to restrict ourselves to linear functions. Since normed spaces are metric spaces with the induced metric and since continuous functions in metric spaces are more important than uncontinuous ones, the important functions between linear spaces are *continuous linear transformations* (*jatkuvat lineaarikuvaukset*).

If X and Y are normed spaces, we shall consider transformations of the type $T : X \longrightarrow Y$. The symbol $||\cdot||$ will stand for the norm in both X and Y as it is usually easy to determine which space an element is in and therefore, implicitly, to which norm we are referring to.

Before looking at examples of continuous linear transformations, it is convenient to give alternative characterizations of continuity:

Lemma 6.1 *Let X and Y be normed linear spaces and let $T : X \longrightarrow Y$ be a linear transformation. Then the following are equivalent:*

(a) T is uniformly continuous,

(b) T is continuous,

(c) T is continuous at 0 (the zero-element in X),

(d) there exists a $k > 0$ such that $||T(x)|| \leq k$ for all $x \in X$ such that $||x|| \leq 1$,

(e) there exists a $k > 0$ such that $||T(x)|| \leq k||x||$ for all $x \in X$.

Example. The transformation $T : C_{\mathbb{F}}[0,1] \longrightarrow \mathbb{F}$ defined by $T(f) = f(0)$ is linear and continuous.

Our next lemma will be used to check that the examples of linear transformations below are well-defined.

Lemma 6.2 *If $\{c_n\} \in \ell^\infty(\mathbb{F})$ and $\{x_n\} \in \ell^p(\mathbb{F})$, $1 \leq p < \infty$, then $\{c_n x_n\} \in \ell^p(\mathbb{F})$ and*

$$\sum_{n=1}^{\infty} |c_n x_n|^p \leq ||\{c_n\}||_\infty^p \sum_{n=1}^{\infty} |x_n|^p. \tag{6.1}$$

Example. If $\{c_n\} \in \ell^\infty(\mathbb{F})$, then the transformation $T : \ell^1(\mathbb{F}) \longrightarrow \mathbb{F}$ given by

$$T(\{x_n\}) = \sum_{n=1}^{\infty} c_n x_n$$

is linear and continuous.

Example. If $\{c_n\} \in \ell^\infty(\mathbb{F})$, then the transformation $T : \ell^p(\mathbb{F}) \longrightarrow \ell^p(\mathbb{F})$, $1 \leq p < \infty$, defined by $T(\{x_n\}) = \{c_n x_n\}$ is linear and continuous.

Transformations satisfying condition (e) of Lemma 6.1 seem to be important. Such transformations have also another name:

Definition 6.3 Let X and Y be normed linear spaces and let $T : X \longrightarrow Y$ be a linear transformation. T is said to be bounded if there exists a $k > 0$ such that $||T(x)|| \leq k||x||$ for all $x \in X$.

By Lemma 6.1 we can use the words *continuous* and *bounded* interchangeably for linear transformations. Note, however, that this is a different use of the word bounded from that used for functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 6.4 Let X and Y be normed linear spaces. The set of all continuous linear transformations from X to Y is denoted by $B(X, Y)$. Elements of $B(X, Y)$ are also called bounded linear operators or linear operators or sometimes just operators.

Remark. If X and Y are normed linear spaces, then $B(X, Y) \subsetneq L(X, Y)$.

The examples presented so far may give the impression that all linear transformations are continuous. Unfortunately, this is not the case:

Example. Let $\mathcal{P}$ be the linear subspace of $C_{\mathbb{R}}[0, 1]$ consisting of all polynomials. Let $T : \mathcal{P} \longrightarrow \mathcal{P}$ be a transformation given by $T(p) = p'$, where p' is the derivative of p. Then T is linear but not continuous.

The space $\mathcal{P}$ in the example above was not finite-dimensional, so it is natural to ask: *Are all linear transformations between finite-dimensional normed spaces continuous?*

Theorem 6.5 *Let X be a finite-dimensional normed linear space, let Y be any normed linear space and let $T : X \longrightarrow Y$ be a linear transformation. Then T is continuous.*

If the domain of definition of a linear transformation is finite-dimensional then the transformation is continuous by Theorem 6.5. On the other hand, if the range is finite-dimensional instead, then the transformation need not be continuous:

Example. Let $\mathcal{P}$ be the linear subspace of $C_{\mathbb{R}}[0, 1]$ consisting of all polynomials. If $T : \mathcal{P} \longrightarrow \mathbb{R}$ is a transformation defined by $T(p) = p'(1)$, where p' is the derivative of p, then T is linear but not continuous.

Finally, we give an elementary property valid for continuous linear transformations:

Theorem 6.6 *If X and Y are normed linear spaces and $T : X \longrightarrow Y$ is a continuous linear transformation then $\mathsf{Ker}(T)$ (kernel, ydin) is closed.*