Analysis IV
Exercise 2
2004

1. Let ℓ^1 be the set of infinite sequences $x = \{x_1, x_2, \dots\}$, $x_n \in \mathbb{C}$, satisfying $\sum_{n=1}^{\infty} |x_n| < \infty$. Show that ℓ^1 equipped with addition $x + y = \{x_1 + y_1, x_2 + y_2, \dots\}$ and scalar multiplication $\alpha x = \{\alpha x_1, \alpha x_2, \dots\}$, $\alpha \in \mathbb{C}$, is an infinite dimensional vector space over $\mathbb{C}$.

2. Prove Schwarz inequality:

$$\left(\sum_{j=1}^{k} |a_j||b_j|\right)^2 \leq \left(\sum_{j=1}^{k} |a_j|^2\right)\left(\sum_{j=1}^{k} |b_j|^2\right), \quad a_j,\, b_j \in \mathbb{C},\, j = 1,\, \dots,\, k.$$

3. Prove Theorem 1.13, (b) and (c).

4. Prove Theorem 1.17, (b).

5. Let

$$\ell^1 = \Big\{x_n\{x_n\}_{n=1}^{\infty}, x_n \in \mathbb{R}, n = 1, 2, \cdots \;\Big|\; \sum_{n=1}^{\infty} |x_n| < \infty\Big\}.$$

   Show that the mapping $d : \ell^1 \times \ell^1 \to \mathbb{R}$,

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|$$

   is a metric.

6. Show that $\overline{E} = \bigcap_{E \subset F} F$ for closed sets F.