Analysis IV
Exercise 3
2004

1. Let (E, d) be a metric space and let $x \in E$. Define
$$d(x, E) = \inf_{y \in E} d(x, y).$$
Show that $\{x : d(x, E) = 0\} = \overline{E}$.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the metric space (M, d). Prove that there exists $R > 0$ such that $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1, R)$.

3. Let $\{a_n\}$ be a Cauchy sequence in the metric space (M, d). Prove: If the sequence $\{a_n\}$ has a subsequence, which converges to $a \in M$, then also $\{a_n\}$ converges to a.

4. Let X be an infinite set. Let $\mathcal{T}$ consist of $\emptyset$, X and all sets G such that $X \setminus G$ is a finite set. Prove that $(X, \mathcal{T})$ is a topological space.

5. Let $A \subset \mathbb{R}^n$ be a set whose every point has a neighbourhood which includes only a countable number of points of A. Prove that A is countable. (Hint: Lindelöf's covering theorem).

6. Let A be a subset of the topological space X. Prove that
$$x \in \{\text{the cluster points (kasaantumispisteet) of } A\}$$
if and only if $x \in \overline{A \setminus \{x\}}$.