Analysis IV
Exercise 5
2004

1. Prove Theorem 2.6.

2. Let (M, d) be a metric space and let f_n, $g_n \in C_{\mathbb{F}}(M)$ such that $\{f_n\} \to f, \{g_n\} \to g$ uniformly on M. Prove that $\{f_n + g_n\}$ converges uniformly on M. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on M.

3. Let $f_n(x) = \frac{1}{n}$, $0 \leq x \leq n$; $f_n(x) = 0$, $x > n$. Prove that $\{f_n\} \to 0$ uniformly on $[0, \infty[$.

4. Prove that $m^*(A \cup B) = m^*(A)$, if $m^*(B) = 0$.

5. Show that if E_1 and E_2 are measurable, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

6. Let $\{E_i\}$ be a sequence of disjoint measurable sets and A any set. Prove that

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

   (Hint: Consider first the claim for a finite union.)