Answers to Exercise 10

I.

Proof . First we prove that $\|\{x_n\}\|_p = \left(\sum\limits_n |x_n|^p\right)^{1/p}$ is a norm on l^p.

1. $\|\{x_n\}\|_p = \left(\sum\limits_n |x_n|^p\right)^{1/p} \geq 0$;
2. $\|\{x_n\}\|_p = 0 \Longleftrightarrow \left(\sum\limits_n |x_n|^p\right)^{1/p} = 0 \Longleftrightarrow |x_n| = 0$, for all $n, \Longleftrightarrow \{x_n\} = \{0, 0, \cdots\}$ the zero element in l^p;
3. for all $\alpha \in \mathbb{F}$, $\|\alpha\{x_n\}\|_p = \|\{\alpha x_n\}\|_p = \left(\sum\limits_n |\alpha x_n|^p\right)^{1/p} = |\alpha| \left(\sum\limits_n |x_n|^p\right)^{1/p} = |\alpha| \|\{x_n\}\|_p$;
4. according to Minkowski inequality, for any $\{x_n\}, \{y_n\} \in l^p$, $\|\{x_n\} + \{y_n\}\|_p = \|\{x_n + y_n\}\|_p = \left(\sum\limits_n |x_n + y_n|^p\right)^{1/p} \leq \left(\sum\limits_n |x_n|^p\right)^{1/p} + \left(\sum\limits_n |y_n|^p\right)^{1/p} = \|\{x_n\}\|_p + \|\{y_n\}\|_p$.

Next we show that $\|\{x_n\}\|_\infty = \sup\limits_n |x_n|$ is a norm on l^∞.

1. $\|\{x_n\}\|_\infty = \sup\limits_n |x_n| \geq 0$;
2. $\|\{x_n\}\|_\infty = 0 \Longleftrightarrow \sup\limits_n |x_n| = 0 \Longleftrightarrow |x_n| = 0$, for all $n \Longleftrightarrow \{x_n\} = \{0, 0, \cdots\}$, the zero element in l^∞;
3. for any $\alpha \in \mathbb{F}$, $\|\alpha\{x_n\}\|_\infty = \|\{\alpha x_n\}\|_\infty = \sup\limits_n |\alpha x_n| = |\alpha| \sup\limits_n |x_n| = |\alpha| \|\{x_n\}\|_\infty$;
4. for any $\{x_n\}, \{y_n\} \in l^\infty$, $\|\{x_n\} + \{y_n\}\|_\infty = \|\{x_n + y_n\}\|_\infty = \sup\limits_n |x_n + y_n| \leq \sup\limits_n (|x_n| + |y_n|) \leq \sup\limits_n |x_n| + \sup\limits_n |y_n| = \|\{x_n\}\|_\infty + \|\{y_n\}\|_\infty$.

II.

Proof . $\Longrightarrow$ Let $\{x_n\}$ be any Cauchy sequence in the metric space (X, d_2), then according to (b) of theorem 4.7, we know that $\{x_n\}$ is also a Cauchy sequence in the metric space (X, d_1). If (X, d_1) is complete, then $\{x_n\}$ converges to x in the metric space (X, d_1), by (a) of theorem 4.7, we get $\{x_n\}$ converges to x in the metric space (X, d_2), so (X, d_2) is complete.

$\Longleftarrow$ Prove in the same way.

III.

Proof . Since $\mathcal{P}$ be the vector space of polynomials on $[0, 1]$, we choose $p_n = x^n$, for any $n \in \mathbb{N}$. $\|p_n\|_1 = \sup\limits_n \{|x_n| : x \in [0, 1]\} = 1$, for any $n \in \mathbb{N}$, and $\|p_n\|_2 = \int_0^1 |x^n| dx = \frac{1}{n+1}$, for any $n \in \mathbb{N}$.

We can't find a constant $C > 0$, s.t. $C\|p_n\|_1 \leq \|p_n\|_2$ holds for all $n \in \mathbb{N}$, that is for all $n \in \mathbb{N}$, $C \leq \frac{1}{n+1}$, C must be equal to zero. $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent on $\mathcal{P}$.

IV.

Proof . We choose a bounded sequence $f_n = x^n$, $\|f_n\| = \sup\{|f_n(x)| : x \in [0,1]\} = 1 < \infty$.

It is easy to know that the only possible limit function of this sequence

$$f_n(x) \longrightarrow f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1 & x = 1. \end{cases}$$

But we can see that f_n has no converging subsequences in such a normed space, since

$$\|f_n - f\| = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n - 0| = 1, \text{ for any } n \in \mathbb{N}.$$

V.

Proof . According to definition of norm equivalence, we only need to find out two constants $m > 0$ and $M > 0$, such that

$$m\|f\|_1 \leq \|f\|_2 \leq M\|f\|_1,$$

which holds for all $f \in C_{\mathbb{R}}([0,1])$.

In fact, since, for $t \in [0,1]$,

$$1 - t^3 = (1-t)(1+t+t^2)$$

and

$$1 - t \leq 1 - t^3 \leq 3(1-t),$$

we get that

$$\|f\|_1 \leq \|f\|_2 \leq 3\|f\|_1$$

holds for all $f \in C_{\mathbb{R}}([0,1])$.

VI.

Proof .

(i) is false. Recall l^1 is the family of all sequences $\{x_n\}$ in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

Choose $x_n = \frac{1}{n}$, it is easy to show that $\{\frac{1}{n}\}_{n=1}^{\infty} \in c_0$, but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is diverge, which means $\{\frac{1}{n}\}_{n=1}^{\infty} \notin l^1$.

(ii) is true. $\{a_n\}_{n=1}^{\infty} \in l^p$, means

$$\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} < \infty$$

and $\{b_n\}_{n=1}^{\infty} \in l^{\frac{p}{p-1}}$, means

$$\left(\sum_{n=1}^{\infty} |b_n|^{\frac{p}{p-1}}\right)^{1-1/p} < \infty.$$

According to Hölder inequality,

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |b_n|^{\frac{p}{p-1}}\right)^{1-1/p} < \infty.$$

$\{a_n b_n\}_{n=1}^{\infty} \in l^1$.

(iii) is true. For any $x = \{x_n\}_{n=1}^{\infty} \in c$, we choose $y = \{-x_n\}_{n=1}^{\infty}$, it is obvious that $y \in c$, then $x + y = \{x_n + (-x_n)\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty} \in c_0$.