Answers to Exercise 11

I.

Proof . First we prove that $c_{0,0} \subset l^p \subset l^\infty$. Recall the definition of l^p, which is the family of all sequences $\{x_n\}$ in $\mathbb{R}$ such that

$$\Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} < \infty, \qquad 1 \le p < \infty,$$

and the space l^∞ is the family of all sequences $\{x_n\}$ in $\mathbb{R}$ such that

$$\sup_n |x_n| < \infty.$$

Choose an arbitrary sequence $\{x_n\}$ from the space $c_{0,0}$, we know there exists a $N_0 \in \mathbb{N}$, s.t. $x_n = 0$, for any $n > N_0$. Then,

$$\Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} = \Big(\sum_{n=1}^{N} |x_n|^p\Big)^{\frac{1}{p}} < \infty,$$

which shows that $\{x_n\} \in l^p$. If we choose an arbitrary sequence $\{x'_n\}$ from l^p, according to Hölder inequality, for any n

$$|x'_n|^p \le \sum_{n=1}^{\infty} |x'_n|^p = \|\{x'_n\}\|_{l^p}^p < \infty,$$

so

$$\sup_n |x'_n| < \infty,$$

we get $\{x'_n\} \in l^\infty$. Thus $c_{0,0} \subset l^p \subset l^\infty$.

Next we prove that $c_{0,0} \subset c_0 \subset c \subset l^\infty$. Let $\{x_n\} \in c_{0,0}$, then there exists $N \in \mathbb{N}$, s.t. $x_n = 0$, for all $n > N$, which implies $\lim_{n\to\infty} x_n = 0$, so $\{x_n\} \in c_0$. We can get $c_0 \subset c$ directly from the definition of the two spaces. Now let's choose a sequence $\{y_n\}$ arbitrarily from space c, we have $\lim_{n\to\infty} y_n = y \in \mathbb{R}$, then for any $1 > \varepsilon > 0$, there exists $N \in \mathbb{N}$, when $n \ge N$

$$|y_n - y| < \varepsilon < 1,$$

that is to say, for all $n \ge N$, $|y_n| < |y| + 1$, and

$$\sup_n |y_n| = \max\{|y_1|, |y_2|, \cdots, |y_{N-1}|, |y| + 1\} < \infty,$$

$\{y_n\} \in l^\infty$. Thus we have showed $c_{0,0} \subset c_0 \subset c \subset l^\infty$.

II.

Proof . It is easy to check that $\mathcal{B}$ is a vector space over $\mathbb{R}$. For all $f, g \in \mathcal{B}$, and for all $\alpha \in \mathbb{R}$,

(a) $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{x\in[0,1]} (1 - |x|^2)|f'(x)| \ge 0,$

(b) if $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)| = 0$, then $|f(0)| = 0$ and $\sup_{x\in[0,1]} (1-|x|^2)|f'(x)| = 0$, which implies $f(x) = C$. for all $x \in [0,1]$, and since $f(0) = 0$, we get $f(x) = f(0) = 0$, i.e. f is the zero function of space C'; if $f \equiv 0$, it is obviously to see that $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)| = 0$,

(c)

$$
\begin{aligned}
\|\alpha f\|_{\mathcal{B}} &= |\alpha f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|(\alpha f)'(x)| \\
&= |\alpha f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|\alpha||f'(x)| \\
&= |\alpha||f(0)| + |\alpha| \sup_{x\in[0,1]} (1-|x|^2)|f'(x)| \\
&= |\alpha|\big(|f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)|\big) \\
&= |\alpha|\|f\|_{\mathcal{B}}\,,
\end{aligned}
$$

(d)

$$
\begin{aligned}
\|f+g\|_{\mathcal{B}} &= |(f+g)(0)| + \sup_{x\in[0,1]} (1-|x|^2)|(f+g)'(x)| \\
&= |f(0)+g(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)+g'(x)| \\
&\le |f(0)| + |g(0)| + \sup_{x\in[0,1]} (1-|x|^2)(|f'(x)| + |g'(x)|) \\
&\le |f(0)| + |g(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)| + \sup_{x\in[0,1]} (1-|x|^2)|g'(x)| \\
&\le \big(|f(0)| + \sup_{x\in[0,1]} (1-|x|^2)|f'(x)|\big) + \big(|g(0)| + \sup_{x\in[0,1]} (1-|x|^2)|g'(x)|\big) \\
&= \|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}\,.
\end{aligned}
$$

$\mathcal{B}$ is a normed space indeed.

III.

Proof . $X\backslash T = \{y \in X : \ \|y\| > 1\}$, the metric in $X\backslash T$ induced by the norm is defined as:

$$d(y_1, y_2) = \|y_1 - y_2\|, \qquad \forall y_1, y_2 \in X\backslash T.$$

Associated with this metric definition, we can get a topology of space $X\backslash T$, where the open set $B(y,r) = \{x \in X : \|x-y\| < r\}$. For any point $y \in X\backslash T$, there exists a positive number d, s.t. $\|y\| = 1 + d > 1$. For any $x \in B(y, \frac{d}{2})$, since $\|y\| - \|x\| \le \|x-y\| < \frac{d}{2}$, we obtain $\|x\| \ge \|y\| - \frac{d}{2} = 1 + d - \frac{d}{2} = 1 + \frac{d}{2} > 1$, so $x \in X\backslash T$, which means $X\backslash T$ is open, i.e. T is closed.

IV.

Proof . From the fact that Y is the subspace of X, we only need to show $X \subset Y$. Since Y is a subspace, $\vec{0} \in Y$, and since Y is open, there exists $r > 0$ and $B(\vec{0}, r) = \{x \in$

$X : \|x-\vec{0}\|_X < r\} \subset Y$. For any $x \in X$, let $\eta = \frac{r}{2}\cdot\frac{x}{\|x\|_X}$, it is easy to see that $\eta \in B(\vec{0}, r)$, for $\|\eta-\vec{0}\|_X = \|\eta\|_X = \|\frac{r}{2}\cdot\frac{x}{\|x\|_X}\|_X = \frac{r}{2} < r$. Thus we can get $\frac{2\|x\|_X}{r}\cdot\eta = x \in Y$, since Y is a subspace, which means $X \subset Y$.

V.

Proof . Prove that

$$< x, y >= \sum_{j=1}^{n} x_j y_j$$

is an inner product on $\mathbb{R}^n$. For all $x, y, z \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$,

(a) $< x, x >= \sum_{j=1}^{n} x_j x_j = \sum_{j=1}^{n} x_j^2 \geq 0$,

(b) if $< x, x >= \sum_{j=1}^{n} x_j^2 = 0$, then $x_j = 0$ for any $j = 1, 2, \cdots, n$, that is $x = (0, 0, \cdots, 0) = \vec{0}$; if $x = (x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0) = \vec{0}$, $< x, x >= \sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} 0^2 = 0$,

(c) $< \alpha x + \beta y, z >= \sum_{j=1}^{n}(\alpha x_j + \beta y_j) z_j = \sum_{j=1}^{n}(\alpha x_j z_j + \beta y_j z_j) = \sum_{j=1}^{n} \alpha x_j z_j + \sum_{j=1}^{n} \beta y_j z_j = \alpha \sum_{j=1}^{n} x_j z_j + \beta \sum_{j=1}^{n} y_j z_j = \alpha < x, z > + \beta < y, z >$,

(d) $< x, y >= \sum_{j=1}^{n} x_j y_j = \sum_{j=1}^{n} y_j x_j =< y, x >$.

Thus $< x, y >= \sum_{j=1}^{n} x_j y_j$ is an inner product on $\mathbb{R}^n$ indeed.