Answers to Exercise 12

I.

Proof . Check the $< x, y >= \sum_{j=1}^{n} \lambda_j \bar{\mu}_j$ with the definition 5.2 of inner product, for all $x, y, z \in X$, and $\alpha, \beta \in \mathbb{C}$.

(a) $< x, x >= \sum_{j=1}^{n} \lambda_j \bar{\lambda}_j = \sum_{j=1}^{n} |\lambda_j|^2 \geq 0$,

(b) $< x, x >= 0 \Longleftrightarrow \sum_{j=1}^{n} |\lambda_j|^2 = 0 \Longleftrightarrow |\lambda_j|^2 = 0,\ \forall j = 1, \cdots, n \Longleftrightarrow \lambda_j = 0,\ \forall j = 1, \cdots, n \Longleftrightarrow x = 0$,

(c) $\alpha x + \beta y = \alpha \sum_{j=1}^{n} \lambda_j e_j + \beta \sum_{j=1}^{n} \mu_j e_j = \sum_{j=1}^{n} (\alpha \lambda_j + \beta \mu_j) e_j$, and let $z = \sum v_j e_j$, then $< \alpha x + \beta y, z >= \sum_{j=1}^{n} (\alpha \lambda_j + \beta \mu_j) \bar{v}_j = \alpha \sum_{j=1}^{n} \lambda_j \bar{v}_j + \beta \sum_{j=1}^{n} \mu_j \bar{v}_j = \alpha < x, z > + \beta < y, z >$,

(d) $< x, y >= \sum_{j=1}^{n} \lambda_j \bar{\mu}_j = \sum_{j=1}^{n} \overline{\bar{\lambda}_j \mu_j} = \overline{\sum_{j=1}^{n} \bar{\lambda}_j \mu_j} = \overline{\sum_{j=1}^{n} \mu_j \bar{\lambda}_j} = \overline{< y, x >}$.

Thus $< x, y >= \sum_{j=1}^{n} \lambda_j \bar{\mu}_j$ is an inner product indeed.

II.

Proof . If $a = \{a_n\}, b = \{b_n\} \in l^2$, then according to Hölder inequality, see (3.4) on page 22,

$$\sum_{n=1}^{\infty} |a_n b_n| \leq (\sum_{n=1}^{\infty} |a_n|^2)^{\frac{1}{2}} (\sum_{n=1}^{\infty} |b_n|^2)^{\frac{1}{2}} < \infty,$$

which showes that $\{a_n b_n\} \in l^1$. Next, let's check that $< a, b >= \sum_{n=1}^{\infty} a_n b_n$ is an inner product on l^2, for all $a, b, c \in l^2$, and $\alpha, \beta \in \mathbb{R}$,

(a) $< a, a >= \sum_{n=1}^{\infty} a_n^2 \geq 0$,

(b) $< a, a >= \sum_{n=1}^{\infty} a_n^2 = 0 \Longleftrightarrow a_n^2 = 0,\ \forall n = 1, 2, \cdots \Longleftrightarrow a = \{0, 0, \cdots\}$, the zero vector in space l^2,

(c) $\alpha a + \beta b = \{\alpha a_n + \beta b_n\}_{n=1}^{\infty}$, since we have proved that for any $a = \{a_n\}, b = \{b_n\} \in l^2$, $\{a_n b_n\} \in l^1$, with this fact, we have $< \alpha a + \beta b, c >= \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) c_n = \sum_{n=1}^{\infty} (\alpha a_n c_n + \beta b_n c_n) = \sum_{n=1}^{\infty} \alpha a_n c_n + \sum_{n=1}^{\infty} \beta b_n c_n = \alpha \sum_{n=1}^{\infty} a_n c_n + \beta \sum_{n=1}^{\infty} b_n c_n = \alpha < a, c > + \beta < b, c >$,

(d) $< a,b >= \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} b_n a_n =< b,a >$.

So, $< a,b >= \sum_{n=1}^{\infty} a_n b_n$ is an inner product on l^2 indeed.

III.

Proof . For any $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in Z$, and any $\alpha,\beta \in \mathbb{C}$,

(a) $< (x,y),(x,y) >=< x,x >_1 + < y,y >_2 \geq 0$,

(b) $< (x,y),(x,y) >=< x,x >_1 + < y,y >_2 = 0 \Longleftrightarrow < x,x >_1 = 0$ and $< y,y >_2 = 0 \Longleftrightarrow$ $x=0$ and $y=0 \Longleftrightarrow (x,y) = (0,0)$,

(c)

$$\begin{aligned}
& < \alpha(x_1,y_1) + \beta(x_2,y_2),(x_3,y_3) > \\
= & < (\alpha x_1,\alpha y_1) + (\beta x_2,\beta y_2),(x_3,y_3) > \\
= & < (\alpha x_1 + \beta x_2,\alpha y_1 + \beta y_2),(x_3,y_3) > \\
= & < \alpha x_1 + \beta x_2, x_3 >_1 + < \alpha y_1 + \beta y_2, y_3 >_2 \\
= & \alpha < x_1,x_3 >_1 + \beta < x_2,x_3 >_1 + \alpha < y_1,y_3 >_2 + \beta < y_2,y_3 >_2 \\
= & \alpha(< x_1,x_3 >_1 + < y_1,y_3 >_2) + \beta(< x_2,x_3 >_1 + < y_2,y_3 >_2) \\
= & \alpha < (x_1,y_1),(x_3,y_3) > + \beta < (x_2,y_2),(x_3,y_3) >,
\end{aligned}$$

(d) Since real vector space is the special case of complex, here, we only need to think about the complex case. $< (x_1,y_1),(x_2,y_2) >=< x_1,x_2 >_1 + < y_1,y_2 >_2 = \overline{< x_2,x_1 >}_1 + \overline{< y_2,y_1 >}_2 = \overline{< (x_2,y_2),(x_1,y_1) >}$.

Thus $< (x,y),(u,v) >=< x,u >_1 + < y,v >_2$ is an inner product on Z.

IV.

Proof . Apply Lemma 5.4 (c),

(a)

$$\begin{aligned}
& < u+v,x+y > - < u-v,x-y > \\
= & (< u,x > + < u,y > + < v,x > + < v,y >) \\
& -(< u,x > + (-1) < u,y > + (-1) < v,x > + (-1)^2 < v,y >) \\
= & 2 < u,y > + 2 < v,x >,
\end{aligned}$$

(b)

$$\begin{aligned}
& i < u+iv,x+iy > - i < u-iv,x-iy > \\
= & i(< u,x > - i < u,y > + i < v,x > + i(-i) < v,y >) \\
& -i(< u,x > + i < u,y > - i < v,x > + (-i)i < v,y >) \\
= & i < u,x > + < u,y > - < v,x > + i < v,y > \\
& -i < u,x > + < u,y > - < v,x > - i < v,y > \\
= & 2 < u,y > - 2 < v,x > .
\end{aligned}$$

So $< u+v, x+y > - < u-v, x-y > +i < u+iv, x+iy > -i < u-iv, x-iy >=$
$2 < u, y > +2 < v, x > +2 < u, y > -2 < v, x >= 4 < u, y > .$

V.

Proof . In the inner product space X, for all $x, y \in X$,

(a) Apply Lemma 5.4 (c),

$$\begin{aligned} & \|x+y\|^2 + \|x-y\|^2 \\ = & < x+y, x+y > + < x-y, x-y >= (< x, x > + < x, y > + < y, x > + < y, y >) \\ & +(< x, x > - < x, y > - < y, x > + < y, y >) \\ = & 2 < x, x > +2 < y, y >= 2(\|x\|^2 + \|y\|^2), \end{aligned}$$

(b) for real X,

$$\begin{aligned} & \|x+y\|^2 - \|x-y\|^2 = (< x, x > +2 < x, y > + < y, y >) \\ & -(< x, x > -2 < x, y > + < y, y >) = 4 < x, y >, \end{aligned}$$

(c) for complex X,

$$\begin{aligned} & \|x+y\|^2 - \|x-y\|^2 = (< x, x > + < x, y > + < y, x > + < y, y >) \\ & -(< x, x > - < x, y > - < y, x > + < y, y >) = 2 < x, y > +2 < y, x >, \end{aligned}$$

and

$$\begin{aligned} & i\|x+iy\|^2 - i\|x-iy\|^2 \\ = & i(< x+iy, x+iy >) - i(< x-iy, x-iy >) \\ = & i(< x, x > -i < x, y > +i < y, x > + < y, y >) \\ & -i(< x, x > +i < x, y > -i < y, x > + < y, y >) \\ = & i < x, x > + < x, y > - < y, x > +i < y, y > \\ & -i < x, x > + < x, y > - < y, x > -i < y, y > \\ = & 2 < x, y > -2 < y, x >, \end{aligned}$$

so

$$\begin{aligned} & \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \\ = & 2 < x, y > +2 < y, x > +2 < x, y > -2 < y, x >= 4 < x, y > . \end{aligned}$$