Answers to Exercise 13

I.

Proof .

$$|T(f)| = \left|\int_0^1 f(x)dx\right| \leq \int_0^1 |f(x)|dx \leq \sup\left\{|f(y)| : y \in [0,1]\right\}\int_0^1 dx = \|f\|\int_0^1 dx = \|f\|,$$

according to Lemma 6.1, we know that T is continuous.

II.

Proof . $h \in L^\infty[0,1]$, we know

$$ess\sup h = \inf\{b|\, h(x) \leq b \text{ a.e. in } [0,1]\} < \infty,$$

it means there exist a number $b \in \mathbb{R}$, and a set $E \subseteq [0,1]$, s.t. $\forall x \in [0,1]\backslash E,\ |h(x)| \leq b$, and $m(E) = 0$. Since $f \in L^2[0,1]$,

(a)

$$\begin{aligned}\int_{[0,1]} |fh|^2 dm &= \int_{[0,1]} |f|^2|h|^2 dm = \int_{[0,1]\backslash E} |f|^2|h|^2 dm + \int_E |f|^2|h|^2 dm \\ &= \int_{[0,1]\backslash E} |f|^2|h|^2 dm \leq b^2 \int_{[0,1]\backslash E} |f|^2 dm \leq b^2 \int_{[0,1]} |f|^2 dm < \infty.\end{aligned}$$

We know that $fh \in L^2[0,1]$.

(b) $T : L^2([0,1]) \longrightarrow L^2([0,1])$, $T(f) = hf$, from case (a), we have

$$\|T(f)\|^2 = \int_{[0,1]} |hf|^2 dm \leq b^2 \int_{[0,1]} |f|^2 dm = b^2\|f\|^2,$$

thus $\|T(f)\| \leq b\|f\|$, from Lemma 6.1, we know that T is continuous.

III.

Proof . For fixed $y \in \mathcal{H}$, there exist a positive number c, s.t. $\|y\| = c$, and f is defined by $f(x) =< x, y >$, then according to Lemma 5.5, for any $\varepsilon > 0$, and for any two points $x, x' \in \mathcal{H}$, when $\|x - x'\| \leq \frac{\varepsilon}{c}$,

$$|f(x) - f(x')| = |< x, y > - < x', y > | = | < x - x', y > | \leq \|x - x'\|\|y\| = c\|x - x'\| < \varepsilon,$$

which means f is continuous.

IV.

Proof .

Since $\{x_1, x_2, x_3, x_4, \cdots\} \in l^2$, we have

$$\sum_{n=1}^\infty |x_n|^2 < \infty,$$

for convenience, we denote $\{y_1, y_2, y_3, y_4, \cdots\} = \{0, 4x_1, x_2, 4x_3, x_4, \cdots\}$,

(a)

$$\begin{aligned}\sum_{n=1}^{\infty}|y_n|^2 &= \sum_{n=1}^{\infty}|y_{2n-1}|^2+\sum_{n=1}^{\infty}|y_{2n}|^2\\ &= \sum_{n=1}^{\infty}|x_{2n}|^2+\sum_{n=1}^{\infty}|4x_{2n-1}|^2\\ &\leq \sum_{n=1}^{\infty}|x_n|^2+16\sum_{n=1}^{\infty}|x_n|^2=17\sum_{n=1}^{\infty}|x_n|^2<\infty.\end{aligned}$$

It means $\{y_1, y_2, y_3, y_4, \cdots\} \in l^2$.

(b) From case (a), we have

$$\|T(\{x_n\})\|^2=\sum_{n=1}^{\infty}|y_n|^2\leq 17\sum_{n=1}^{\infty}|x_n|^2=17\|\{x_n\}\|^2,$$

which means $\|T(\{x_n\})\| \leq \sqrt{17}\|\{x_n\}\|$, apply Lemma 6.1 again, we know T is continuous.