Answers to Exercise 9

I.

Proof . Let $\{\{x_{n,k}\}\} \subset l^\infty$ be a Cauchy sequence, which means $\forall \varepsilon > 0$, for convenience, we let $\varepsilon < 1$, there exists $N \in \mathbb{N}$, for any $n, m \geq N$,

$$d_{l^\infty}(\{x_{n,k}\}, \{x_{m,k}\}) < \varepsilon.$$

Then we have

$$|x_{n,k} - x_{m,k}| \leq \sup_k |x_{n,k} - x_{m,k}| = d_{l^\infty}(\{x_{n,k}\}, \{x_{m,k}\}) < \varepsilon.$$

Since $\mathbb{R}$ is complete, for any k, there must exist x_k s.t.

$$\lim_{n\to\infty} x_{n,k} = x_k.$$

Thus we can get a sequence $\{x_k\}_{k=1}^\infty$.

For $|x_{n,k} - x_{m,k}| < \varepsilon$, we let $m \to \infty$, then

$$|x_{n,k} - x_k| \leq \varepsilon,\ \forall k \in \mathbb{N}, \text{and } n \geq N,$$

thus for all $n \geq N$,

$$d_{l^\infty}(\{x_{n,k}\}, \{x_k\}) = \sup_k |x_{n,k} - x_k| \leq \varepsilon < 1.$$

Next we prove that $\{x_k\}_{k=1}^\infty \in l^\infty$.

$$\begin{aligned} \sup_k |x_k| &= \sup_k |x_k - x_{N,k} + x_{N,k}| \leq \sup_k(|x_k - x_{N,k}| + |x_{N,k}|) \\ &\leq \sup_k |x_k - x_{N,k}| + \sup_k |x_{N,k}| < 1 + \|x_{N,k}\|_{l^\infty} < \infty. \end{aligned}$$

So $\{x_k\} \in l^\infty$, which means Cauchy sequence $\{x_{n,k}\}$ converges to a sequence in l^∞ in the metric d_{l^∞}.

II.

Proof .

a) For any $\varepsilon > 0$, we choose $N = [\varepsilon^{-\frac{1}{2}}] + 1$, when $n \geq N$,

$$|f_n(x) - 0| \leq \frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{\left(\varepsilon^{-\frac{1}{2}}\right)^2} = \varepsilon.$$

$\{f_n(x)\}$ converges pointwise to zero.

b) For any $n \in \mathbb{N}$, we choose $\varepsilon = \frac{1}{2n^2} > 0$, and choose $\delta = 1 > 0$,

$$m(\{x | \, |f_n(x) - 0| \geq \frac{1}{2n^2}\}) = m([-n, n]) = 2n > 1.$$

So $\{f_n\}$ does not converge in the measure m.

c)

$$\begin{aligned} d_{L^p}(f_n,0) &= \Big(\int |f_n|^p dm\Big)^{\frac{1}{p}} = \frac{1}{n^2}\Big(\int \chi_{[-n,n]} dm\Big)^{\frac{1}{p}} = \frac{1}{n^2}\big(m([-n,n])\big)^{\frac{1}{p}} = \frac{(2n)^{\frac{1}{p}}}{n^2} \\ &= 2^{\frac{1}{p}} n^{\frac{1}{p}-2} \longrightarrow 0 \text{ when } 1 \le p \le \infty. \end{aligned}$$

$\{f_n\}$ converges with respect to d_{L^p}-metric when $1 \le p \le \infty$.

III.

Proof . $\forall (x_1,y_1) \in Z$, $(x_2,y_2) \in Z$, and $\alpha \in \mathbb{F}$, we define $(x_1,y_1)+(x_2,y_2) = (x_1 + x_2, y_1+y_2)$, and $\alpha(x,y) = (\alpha x, \alpha y)$.

First to show that $Z = X \times Y$ is a vector space.

(a) $\forall (x_i,y_i) \in Z$, $i = 1,2,3$,

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2) = (x_2+x_1, y_2+y_1) = (x_2,y_2)+(x_1,y_1),$$

and

$$\begin{aligned} (x_1,y_1)+\big((x_2,y_2)+(x_3,y_3)\big) &= (x_1,y_1)+(x_2+x_3, y_2+y_3) = (x_1+x_2+x_3, y_1+y_2+y_3) \\ &= (x_1+x_2, y_1+y_2)+(x_3,y_3) = \big((x_1,y_1)+(x_2,y_2)\big)+(x_3,y_3). \end{aligned}$$

(b) $\vec{0} = (\vec{0}_X, \vec{0}_Y) \in Z$, s.t. $\forall (x,y) \in Z$, $(\vec{0}_X,\vec{0}_Y)+(x,y) = ((\vec{0}_X + x, \vec{0}_Y + y) = (x,y)$.

(c) $\forall (x,y) \in Z$, there exists unique $(-x,-y) \in Z$, where $-x$ is the inverse vector of x in space X, $x+(-x) = \vec{0}_X$, and $-y$ is the inverse vector of y in space Y, $y+(-y) = \vec{0}_Y$. Then $(x,y)+(-x,-y) = \big(x+(-x), y+(-y)\big) = (\vec{0}_X,\vec{0}_Y) = \vec{0}$.

(d) $1 \in \mathbb{F}$, $1 \cdot (x,y) = (1\cdot x, 1\cdot y) = (x,y)$, $\forall \alpha,\beta \in \mathbb{F}$, $\alpha\big(\beta(x,y)\big) = \alpha(\beta x, \beta y) = (\alpha\beta x, \alpha\beta y) = (\alpha\beta)(x,y)$.

(e)

$$\begin{aligned} \alpha\big((x_1,y_1)+(x_2,y_2)\big) &= \alpha(x_1+x_2, y_1+y_2) = (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2) \\ &= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2) \\ &= \alpha(x_1,y_1)+\alpha(x_2,y_2), \end{aligned}$$

and

$$\begin{aligned} (\alpha+\beta)(x,y) &= \big((\alpha+\beta)x, (\alpha+\beta)y\big) = (\alpha x + \beta x, \alpha y + \beta y) \\ &= (\alpha x, \alpha y) + (\beta x, \beta y) = \alpha(x,y)+\beta(x,y). \end{aligned}$$

So $Z = X \times Y$ is a vector space indeed.

To prove that $Z = X \times Y$ is a normed space with the norm defined as this: $\forall (x,y) \in Z = X \times Y$, $x \in X$, $y \in Y$, the norm $\|(x,y)\| = \|x\|_1 + \|y\|_2$.

(a) Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms of X, Y respectively, $\|(x,y)\| = \|x\|_1 + \|y\|_2 \ge 0$.

(b) If some vector $(x, y) \in Z$, s.t. $\|(x, y)\| = 0$, from the definition, $\|(x, y)\| = \|x\|_1 + \|y\|_2 = 0$, it must be $\|x\|_1 = 0$ and $\|y\|_2 = 0$, which mean $x = \vec{0}_X$, and $y = \vec{0}_Y$, that is $(x, y) = (\vec{0}_X, \vec{0}_Y) = \vec{0}$ be the zero vector of Z. On the other side, for zero vector $(\vec{0}_X, \vec{0}_Y) \in Z$, $\|(\vec{0}_X, \vec{0}_Y)\| = \|\vec{0}_X\|_1 + \|\vec{0}_Y\|_2 = 0 + 0 = 0$.

(c) For any $\alpha \in \mathbb{F}$, and any $(x, y) \in Z$, $\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \|\alpha x\|_1 + \|\alpha y\|_2 = |\alpha|\|x\|_1 + |\alpha|\|y\|_2 = |\alpha|(\|x\|_1 + \|y\|_2) = |\alpha|\|(x, y)\|$.

(d) For any $(x, y) \in Z$, $(x', y') \in Z$,

$$\begin{aligned} \|(x, y) + (x', y')\| &= \|(x + x', y + y')\| = \|x + x'\|_1 + \|y + y'\|_2 \\ &\leq (\|x\|_1 + \|x'\|_1) + (\|y\|_2 + \|y'\|_2) \\ &= (\|x\|_1 + \|y\|_2) + (\|x'\|_1 + \|y'\|_2) \\ &= \|(x, y)\| + \|(x', y')\|. \end{aligned}$$

IV.

Proof . The norm in space $C_{\mathbb{R}}([0, 1])$ is defined as:

$$\|f\| = \sup\{|f(x)|\, |x \in [0, 1]\}.$$

For function $f_n(x) = x^n$, $\|f_n(x)\| = \sup\{|x^n|\, |x \in [0, 1]\} = 1$.

The norm in space $L^1([0, 1])$ is defined as:

$$\|f\|_p = \int_0^1 |f| dm.$$

So

$$\|f_n(x)\| = \int_{[0,1]} |f_n(x)| dm = \int_{[0,1]} x^n dm = \int_0^1 x^n dx = \frac{1}{n + 1}.$$