Answers to Exercise 1

I.

Proof. Since $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in the space $\mathbb{R}^n$, it must be the maximum independent vector group. So for any $x \in \mathbb{R}^n$, $\{x, x_1, \cdots, x_n\}$ will be the dependent set of vectors, and any vector of this set can be represented by the others, so there are $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, such that $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$. If there also exist $\lambda'_1, \cdots \lambda'_n \in \mathbb{R}$, such that $x = \lambda'_n x_1 + \cdots \lambda'_n x_n$, we have

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = \lambda'_n x_1 + \cdots \lambda'_n x_n \,,$$

that is $(\lambda_1 - \lambda'_1)x_1 + \cdots + (\lambda_n - \lambda'_n)x_n = 0$. Since $x_1, \cdots, x_n$ are independent, we obtain $\lambda_1 = \lambda'_1, \cdots, \lambda_n = \lambda'_n$. So, every $x \in \mathbb{R}^n$ has at most one representation of the form $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

II.

{That is to show $\{e_1, \cdots, e_n\}$ is the maximum independent vector group}.

Proof. $e_i = \{0, \cdots, 1, \cdots, 0\}$, it is easy to see that $e_1, \cdots, e_n$ is independent. For any $\bar{a} \in \mathbb{R}^n, \bar{a} = \{a_1, \cdots, a_n\} = a_1 e_1 + \cdots + a_n e_n$, so there exist a set of number $1, -a_1, \cdots, -a_n$, such that the linear assembler of $\bar{a}, e_1, \cdots, e_n$ equals to zero, it means $\bar{a}, e_1, \cdots, e_n$ is linear dependent, from the arbitrariness of $\bar{a}$, we know $\{e_1, \cdots, e_n\}$ is the maximum independent group of $\mathbb{R}^n$, so $\{e_1, \cdots, e_n\}$ is not a strict subset of any linearly independent set of $\mathbb{R}^n$.

III.

$\Big\{$Definition of vector space: $V \times V \longrightarrow V, F \times F \longrightarrow V$. For any $\alpha, \beta \in F$, and $x, y, z \in V$

1. $x + y = y + x, x + (y + z) = (x + y) + z$;
2. $\exists \bar{0} \in V$, such that $\bar{0} + x = x + \bar{0} = x$;
3. for any $x \in V, \exists - x \in V$, such that $x + (-x) = \bar{0}$;
4. $1 \cdot x = x, \alpha(\beta x) = (\alpha\beta)x$;
5. $\alpha(x + y) = \alpha x + \alpha y, (\alpha + \beta)x = \alpha x + \beta x$.

Definition 1.5 V : F-vector space, $f : S \longrightarrow V, f, g \in F(S, V)$, and $f + g \in F(S, V), \alpha f \in F(S, V), (f + g)(x) = f(x) + g(x), (\alpha f)(x) = \alpha f(x)$, for any $x \in S.\Big\}$

Proof. For any $f, g, h \in F(S, V)$ and $\alpha, \beta \in F, x \in S$.

1. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x) \in V$, that is $f + g = g + f$; $[f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + g(x) + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x)$, that is $f + (g + h) = (f + g) + h$;
2. $\exists \bar{0} \in F(S, V), (f + \bar{0})(x) = f(x) + \bar{0}(x) = f(x) + 0 = 0 + f(x) = \bar{0} + f(x) = (\bar{0} + f)(x), \bar{0}(x) = 0 \in V$, for $x \in S$;
3. for any $f \in F(S, V), -1 \cdot f = -f \in F(S, V)$, and $[f + (-f)](x) = f(x) + (-f)(x) = 0$, that is $f + (-f) = \bar{0}$;

4. $1 \in F, (1 \cdot f)(x) = 1 \cdot f(x) = f(x)$, so $1 \cdot f = f$; $\alpha(\beta f)(x) = \alpha \cdot \beta f(x) = (\alpha\beta) f(x) \in V$, so $\alpha(\beta f) = (\alpha\beta) f$;

5. $\alpha(f+g)(x) = \alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x) \in V$, so $\alpha(f+g) = \alpha f + \alpha g$; $(\alpha+\beta) \cdot f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x) \in V$, so $(\alpha+\beta) \cdot f = \alpha f + \beta f$.

Thus $F(S, V)$ is F-vector space.

IV.

Proof. $\{V : F\text{-vector space}, U \subset V, V, U \text{ linear.}\}$ $x_i \in U, a_i \in F$, so $a_i x_i \in U, i = 1, \cdots, p$. Because U is a linear subspace, $a_1x_1 + a_2x_2 \in U$, so if $a_1x_1 + a_2x_2 + \cdots + a_kx_k \in U, 1 < k < p$, then $a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1} = (a_1x_1 + a_2x_2 + \cdots + a_kx_k) + a_{k+1}x_{k+1} \in U$, so $a_1x_1 + \cdots + a_px_p \in U$.

V.

Proof. $\Leftarrow$ is obvious.

$\Rightarrow$ If $U_1 \subset U_2$ and $U_2 \subset U_1$, there exists at least one element $x \in U_1$, but $x \in U_2$, and at least one $y \in U_2$, but $y \in U_1$. Since $U_1 \cup U_2$ is the subspace of $V, x + y \in U_1 \cup U_2$, then $x + y \in U_1$ or $x + y \in U_2$. If $x + y \in U_1$, then $x + y + (-x) = y \in U_1$, since $-x \in U_1$, which produces a contradiction. It is the same to U_2. So the assumption is wrong.

VI.

$\Big\{$Linear subspace: V : F-vector space, $U \subset V$. If for any $x, y \in U, \alpha, \beta \in F \Rightarrow \alpha x + \beta y \in U\Big\}$.

Proof. Since U_i is the linear subspace of $V, \bar{0} \in U_i$ for any i, $U = \bigcap_{i \in I} U_i \neq \emptyset$, and it is obvious that $U \subset V$. For any $x, y \in U = \bigcap_{i \in I} U_i$ and $\alpha, \beta \in F$, $x, y \in U_i$, for all $i \in I$, then $\alpha x + \beta y \in U_i$ for all $i \in I$, so $\alpha x + \beta y \in \bigcap_{i \in I} U_i = U$, so U is also a linear subspace of V.