Analysis IV
Exercise 10
2004

1. Proof the third example on page 25.

2. Prove Theorem 4.7 (c).

3. Let $\mathcal{P}$ be the vector space of polynomials defined on $[0,1]$. As $\mathcal{P}$ is a linear subspace of $\mathcal{C}_{\mathbb{F}}([0,1])$ it has a norm $\|p\|_1 = \sup\{|p(x)| : x \in [0,1]\}$ and as $\mathcal{P}$ is a linear subspace of $L^1[0,1]$ it has another norm $\|p\|_2 = \int_0^1 |p(x)|dx$. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent on $\mathcal{P}$.

4. Find in the space $\mathcal{C}_{\mathbb{R}}([0,1])$ equipped with the standard norm
$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}$$
a bounded sequence which has not converging subsequences.

5. Show that in the space $\mathcal{C}_{\mathbb{R}}([0,1])$ the norms
$$\|f\|_1 = \int_0^1 (1-t)|f(t)|dt$$
and
$$\|f\|_2 = \int_0^1 (1-t^3)|f(t)|dt$$
are equivalent.

6. Introduce
$$c = \left\{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{R}, \exists \lim_{n\to\infty} x_n\right\}$$
and
$$c_0 = \left\{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{R}, \lim_{n\to\infty} x_n = 0\right\}.$$
Prove true or false:

   (i) $c_0 \subset l^1$,

   (ii) If $\{a_n\}_{n=1}^{\infty} \in l^p$ and $\{b_n\}_{n=1}^{\infty} \in l^{\frac{p}{p-1}}$, where $1 < p < \infty$, then $\{a_n b_n\}_{n=1}^{\infty} \in l^1$,

   (iii) If $x \in c$, then there exists such $y \in c$ that $x + y \in c_0$.