Answers to Exercise 2

I.

Proof . For any $\alpha, \beta \in \mathbb{C}$, and any $x, y, z \in l^1, x = \{x_1, x_2, \cdots\}, y = \{y_1, y_2, \cdots\}, z = \{z_1, z_2, \cdots\}$, we have the followings:

1. $x + y = \{x_1 + y_1, x_2 + y_2, \cdots\} = \{y_1 + x_1, y_2 + x_2, \cdots\} = y + x$; and

$$\begin{aligned} x + (y + z) &= \{x_1, x_2, \cdots\} + \{y_1 + z_1, y_2 + z_2, \cdots\} \\ &= \{x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \cdots\} \\ &= \{x_1 + y_1 + z_1, x_2 + y_2 + z_2, \cdots\} \\ &= \{(x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \cdots\} \\ &= \{x_1 + y_1, x_2 + y_2, \cdots\} + \{z_1, z_2, \cdots\} \\ &= (x + y) + z \end{aligned}$$

2. $\bar{0} = \{0, 0, \cdots\} \in l^1$, such that $x + \bar{0} = \{x_1 + 0, x_2 + 0, \cdots\} = \{x_1, x_2, \cdots\} = x$;

3. $-1 \cdot x = -x = \{-x_1, -x_2, \cdots\} \in l^1$, such that $x + (-x) = \{x_1 + (-x_1), x_2 + (-x_2), \cdots\} = \{0, 0, \cdots\} = \bar{0}$;

4. $1 \cdot x = \{1 \cdot x_1, 1 \cdot x_2, \cdots\} = \{x_1, x_2, \cdots\} = x$; and $\alpha(\beta x) = \alpha\{\beta x_1, \beta x_2, \cdots\} = \alpha \cdot \beta\{x_1, x_2, \cdots\} = \alpha\beta x$;

5.

$$\begin{aligned} \alpha(x + y) &= \alpha\{x_1 + y_1, x_2 + y_2, \cdots\} \\ &= \{\alpha(x_1 + y_1), \alpha(x_2 + y_2), \cdots\} \\ &= \{\alpha x_1 + \alpha x_2, \cdots\} + \{\alpha y_1 + \alpha y_2, \cdots\} \\ &= \alpha\{x_1, x_2, \cdots\} + \alpha\{y_1, y_2, \cdots\} \\ &= \alpha x + \alpha y \end{aligned}$$

and

$$\begin{aligned} (\alpha + \beta)x &= (\alpha + \beta)\{x_1, x_2, \cdots\} \\ &= \{(\alpha + \beta)x_1, (\alpha + \beta)x_2, \cdots\} \\ &= \{\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \cdots\} \\ &= \{\alpha x_1, \alpha x_2, \cdots\} + \{\beta x_1, \beta x_2, \cdots\} \\ &= \alpha x + \beta x. \end{aligned}$$

So l^1 is a vector space.

Besides, since l^1 is the set of such infinite sequences $\{x_1, x_2, \cdots\}, x_n \in \mathbb{C}$, the basis of l^1 is $\{e_i\}_{i=1}^{\infty}$, where $e_i = \{0, \cdots, 1_i, \cdots\}$, it means any finite set $\{e_i\}_{i=1}^{k}$ is linearly independent for any $k \in \mathbb{N}$, so dim $l^1 \geq k$, for any $k \in \mathbb{N}$, then we get dim $l^1 = \infty$. Thus l^1 is an infinite dimensional vector space.

II.

Proof. When $k = 1, (|a_1||b_1|)^2 \le |a_1|^2|b_1|^2$; If, when $k = m$, the inequality

$$\Big(\sum_{j=1}^{m} |a_j||b_j|\Big)^2 \le \Big(\sum_{j=1}^{m} |a_j|^2\Big)(\sum_{j=1}^{k} |b_j|^2)$$

holds for $1 < m < k$, then, when $k = m + 1$,

$$\begin{aligned}
\Big(\sum_{j=1}^{m+1} |a_j||b_j|\Big)^2 &= \Big(\sum_{j=1}^{m} |a_j||b_j| + |a_{m+1}||b_{m+1}|\Big)^2 \\
&= \Big(\sum_{j=1}^{m} |a_j||b_j|\Big)^2 + 2|a_{m+1}||b_{m+1}|\sum_{j=1}^{m} |a_j||b_j| + |a_{m+1}|^2|b_{m+1}|^2 \\
&\le \Big(\sum_{j=1}^{m} |a_j|^2\Big)\Big(\sum_{j=1}^{m} |b_j|^2\Big) + 2|a_{m+1}||b_{m+1}|\sum_{j=1}^{m} |a_j||b_j| + |a_{m+1}|^2|b_{m+1}|^2 \,.
\end{aligned}$$

Since

$$\sum_{j=1}^{m} |a_j||b_j| \le \sum_{j=1}^{m} |a_j| \sum_{j=1}^{m} |b_j| \,,$$

and since

$$\begin{aligned}
0 &< \Big(|b_{m+1}|\sum_{j=1}^{m} |a_j| - |a_{m+1}|\sum_{j=1}^{m} |b_j|\Big)^2 \\
&= |b_{m+1}|^2\Big(\sum_{j=1}^{m} |a_j|\Big)^2 - 2|a_{m+1}||b_{m+1}|\sum_{j=1}^{m} |a_j|\sum_{j=1}^{m} |b_j| + |a_{m+1}|^2\Big(\sum_{j=1}^{m} |b_j|\Big)^2 \\
&\le |b_{m+1}|^2\sum_{j=1}^{m} |a_j|^2 - 2|a_{m+1}||b_{m+1}|\sum_{j=1}^{m} |a_j|\sum_{j=1}^{m} |b_j| + |a_{m+1}|^2\sum_{j=1}^{m} |b_j|^2 \,,
\end{aligned}$$

we get

$$2|a_{m+1}||b_{m+1}|\sum_{j=1}^{m} |a_j|\sum_{j=1}^{m} |b_j| \le |a_{m+1}|^2\sum_{j=1}^{m} |b_j|^2 + |b_{m+1}|^2\sum_{j=1}^{m} |a_j|^2 \,.$$

So

$$\begin{aligned}
\Big(\sum_{j=1}^{m+1} |a_j||b_j|\Big)^2 &\le \Big(\sum_{j=1}^{m} |a_j|^2\Big)\Big(\sum_{j=1}^{m} |b_j|^2\Big) + |a_{m+1}|^2\sum_{j=1}^{m} |b_j|^2 + |b_{m+1}|^2\sum_{j=1}^{m} |a_j|^2 + |a_{m+1}|^2|b_{m+1}|^2 \\
&= \Big(\sum_{j=1}^{m} |a_j|^2 + |a_{m+1}|^2\Big)\Big(\sum_{j=1}^{m} |b_j|^2 + |b_{m+1}|^2\Big) \\
&= \Big(\sum_{j=1}^{m+1} |a_j|^2\Big)\Big(\sum_{j=1}^{m+1} |b_j|^2\Big) \,,
\end{aligned}$$

where $a_j, b_j \in \mathbb{C}, j = 1, \cdots, k$.

By induction, we obtain the Schwarz inequality:

$$\Big(\sum_{j=1}^{k} |a_j||b_j|\Big)^2 \leq \Big(\sum_{j=1}^{k} |a_j|^2\Big)\Big(\sum_{j=1}^{k} |b_j|^2\Big).$$

III.

Proof. $\{x_n\}$ is a convergent sequence in a metric space (M, d) and there exists a unique $x \in M$, such that $\lim\limits_{n\to\infty} x_n = x$.

To show (b): If $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}$, we want to show that $x_{n_k} \longrightarrow x$.

Since $\lim\limits_{n\to\infty} x_n = x$, for any $\varepsilon > 0$, there exist a $N_\varepsilon \in N$, when $n > N_\varepsilon$, $d(x_n, x) < \varepsilon$.

In case of the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, for the above ε and the N_ε, when $n_k > N_\varepsilon$, we will also have $d(x_{n_k}, x) < \varepsilon$, so $\lim\limits_{n_k\to\infty} x_{n_k} = x$.

To show (c): For any $\varepsilon > 0$, there exist $N_\varepsilon \in N$, when $n > N_\varepsilon, d(x_n, x) < \frac{\varepsilon}{2}$. Then for all $m, n > N_\varepsilon$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

so $\{x_n\}$ is a Cauchy sequence.

IV.

Proof. (i). $\{f$ continuous on M (means f is continuous at each point of M) $\Rightarrow \forall$ open set $A \subset N, f^{-1}(A) \subset M$ is open. $\}$

$\forall x \in f^{-1}(A)$, there exists a $y \in A$, s.t. $f(x) = y$. Since A is an open set, there exists a ε_0 s.t. $B_N(y, \varepsilon_0) \subset A$. Since f is continuous, from the definition of continuity, we know that for the above $\varepsilon_0 > 0$, there exists a $\delta > 0$, s.t. $f(B_M(x, \delta)) \subset B_N(y, \varepsilon_0) \subset A$, that is $B_M(x, \delta) \subset f^{-1}(A)$, it means that $f^{-1}(A)$ is open.

$\Leftarrow$: On the contrary, if f is not continuous on M, according to the definition there exists at least one point $x \in M$, s.t. f is not continuous at x, it means there exists a $\varepsilon_0 > 0$, s.t. for any $\delta > 0, f(B_M(x, \delta))$ is not the subset of $B_N(f(x), \varepsilon_0)$. We know $B_N(f(x), \varepsilon_0)$ is an open set of N, and it is obvious that $x \in f^{-1}\Big(B_N(f(x), \varepsilon_0)\Big)$, but, there is no $\delta > 0$, satisfies $f(B_M(x, \delta)) \subset B_N(f(x), \varepsilon_0)$, which means no $\delta > 0$, such that $B_M(x, \delta) \subset f^{-1}\Big(B_N(f(x), \varepsilon)\Big)$, it is a contradictory to the condition that $f^{-1}\Big(B_N(f(x), \varepsilon_0)\Big)$ is open set. So f must be continuous.

(ii). $\{$ f is continuous $\Rightarrow$ for any closed set $\cdots\}$

For any $x \in M\backslash f^{-1}(A), f(x) \notin A, f(x) \in N\backslash A$. Since A is closed, $N\backslash A$ is open, then there exists a $\varepsilon > 0$, s.t. $B_N(f(x), \varepsilon) \subset N\backslash A$, it is obvious that $x \in f^{-1}\Big(B_N(f(x), \varepsilon)\Big)$. From the continuity of f, we know, for the above ε, there exists a $\delta > 0$, such that $f(B_M(x, \delta)) \subset B_N(f(x), \varepsilon)$, with this, we get $f(x') \in N\backslash A$, for any $x' \in B_M(x, \delta)$, thus $x' \in M\backslash f^{-1}(A)$,that is $B_M(x, \delta) \subset M\backslash f^{-1}(A)$. From the arbitrariness of x, we obtain that $M\backslash f^{-1}(A)$ is open, that is $f^{-1}(A)$ is closed.

$\Leftarrow$: We can use the same method as in (i), but now we can also employ the conclusion of (i), for it has been proved. Since $A \subset N$ is closed, $N\backslash A$ is open, and $f^{-1}(A) \subset M$ is closed, $M\backslash f^{-1}(A) = f^{-1}(N\backslash A)$ is open, then according to (i), we immediately get f is continuous.

V.

Proof.$\{d : l^1 \times l^1 \to \mathbb{R}$, for any $x = \{x_n\} \in l^1, y = \{y_n\} \in l^1, d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|$, to show that d is a metric. $\}$

1. $d(x, y) = d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n| \geq 0$;
2. $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n| = 0 \Rightarrow x_n = y_n, n = 1, 2, \cdots$, then $\{x_n\} = \{y_n\}$, that is $x = y$; and if $x = y$, that is $\{x_n\} = \{y_n\}, x_n = y_n, n \in N, d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n| = \sum_{n=1}^{\infty} |x_n - x_n| = 0$;
3. $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n| = \sum_{n=1}^{\infty} |y_n - x_n| = d(\{y_n\}, \{x_n\}) = d(y, x)$;
4. $d(x, z) = d(\{x_n\}, \{z_n\}) = \sum_{n=1}^{\infty} |x_n - z_n| \leq \sum_{n=1}^{\infty} |x_n - y_n + y_n - z_n| \leq \sum_{n=1}^{\infty} (|x_n - y_n| + |y_n - z_n|) = \sum_{n=1}^{\infty} |x_n - y_n| + \sum_{n=1}^{\infty} |y_n - z_n| = d(x, y) + d(y, z)$,

so, d is a metric.

VI.

Proof. First to show that the closure is closed. From the definition of closure,

$$\bar{E} = E \cup E' = E \cup \{\text{all the accumulate points of } E\},$$

accumulate points a of E refers to such kind of points, which satisfy $\{B(a, r) \backslash a\} \bigcap E \neq \emptyset$, for any $r > 0$. We consider the complement of $\bar{E}$, denoted by $\bar{E}^-$, for any point $x \in \bar{E}^-$, there must exist $\delta > 0$, s.t. $B(x, \delta) \cap \bar{E} = \emptyset$, otherwise, if there exists no such δ, then $x \in \bar{E}'$, that means $B(x, \delta) \cap \bar{E} \neq \emptyset$, for any $\delta > 0$, it means there exists $x \neq y \in \bar{E}, y \in B(x, \delta)$, then denote $r_1 = d(x, y)$, and $r_2 = d(y, B(x, \delta))$, choose $r_\delta = \min\{r_1, r_2\}$, we get $B(y, r_\delta) \subset B(x, \delta)$, since $y \in \bar{E}$, there must exist point $z \in E$, s.t. $z \in B(y, r_\delta) \subset B(x, \delta)$, which is impossible since $x \notin \bar{E}$. So from the arbitrariness of x, we know that $\bar{E}^-$ is open, then $\bar{E}$ is closed. That is $\bar{E} \in \{F | E \subset F, F \text{ is closed}\}$, then we obtain $\bigcap_{E \subset F} F \subseteq \bar{E}$.

On the other side, for any $x \in \bar{E}$, $x \in E$, or x is the accumulate point of E, if $x \in E$, then $x \in F$, for any $F \supset E$, so $x \in \bigcap_{E \subset F} F$; if x is the accumulate point of E, we also know $x \in F$, since F is closed, and $E \subset F$, the accumulate point of the subset of F must belong to F, for any closed $F \supset E$, so $x \in \bigcap_{E \subset F} F$. Thus $\bar{E} \subseteq \bigcap_{E \subset F} F$. We finish the proof.