Answers to Exercise 3

I.

Proof . Step 1: to show $\{x : d(x,E) = 0\} \subset \bar{E}$. $\forall x \in \{x : d(x,E) = 0\}, d(x,E) = \inf_{y\in E}(x,y) = 0$. If $x \in E$, it is obvious that $d(x,E) = 0$; if $x \notin E$, we assert that x must be a cluster point of E, if not, there exists $\delta > 0$, such that $B_d(x,\delta) \cap E = \emptyset$, then $d(x,E) \geq \delta > 0$, which is contradictory to the given condition, so x must be a cluster of E.

Step 2: to show $\bar{E} \subset \{x : d(x,E) = 0\}$. $x \in \bar{E}, x \in E$, or x is the cluster of point E, if $x \in E, d(x,E) = 0$; if x is the cluster point of E, then for any $\varepsilon > 0$, there exists $y \in E$, s.t. $y \in B_d(x,\varepsilon)$, that is $d(x,y) < \varepsilon$, for any $\varepsilon > 0$, so $d(x,E) = \inf_{y\in E} d(x,y) = 0$, $x \in \{x : d(x,E) = 0\}$.

Thus $\{x : d(x,E) = 0\} = \bar{E}$.

II.

Proof . $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (M,d). Let $\varepsilon = 1 > 0$, there exists $N^* \in \mathbb{N}$, s.t. $\forall m,n \geq N^*, d(x_m,x_n) < 1$, so if $k \geq N^*$, $d(x_k,x_1) \leq d(x_k,x_{N^*}) + d(x_{N^*},x_1) < 1 + d(x_{N^*},x_1)$; if $K \leq N^*$, denote $b = \max\{d(x_2,x_1), d(x_3,x_1), \cdots, d(x_{N^*},x_1)\}$, $d(x_k,x_1) \leq b$, so there exists $R = max\{b, 1 + d(x_{N^*},x_1)\}$, s.t. $\{x_n\}_{n=1}^{\infty} \subset B_d(x_1,R)$.

III.

Proof . Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}$ and $a_{n_k} \to a$, as $k \to \infty$, then for any $\varepsilon > 0$, there exists $N_{k,\varepsilon} \in \mathbb{N}$, s.t. for any $n_k \geq N_{k,\varepsilon}$, $d(a_{n_k},a) < \frac{\varepsilon}{2}$. Since $\{a_n\}$ is a Cauchy sequence, we know that, for the same $\varepsilon > 0$, there exists $N'_\varepsilon \in \mathbb{N}$, for any $m,n \geq N'_\varepsilon$, $d(a_m,a_n) < \frac{\varepsilon}{2}$. Let $M_\varepsilon = \max(N_{k,\varepsilon}, N'_\varepsilon)$, if $n \geq M_\varepsilon$, then $d(a_n,a) \leq d(a_n,a_{n_k}) + d(a_{n_k},a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, thus we get $\{a_n\} \longrightarrow a$, as $n \longrightarrow \infty$.

IV.

Proof . {Recall the definition of topology: Let X be a set , and Γ is a collection of subsets of X such that:

1. $\emptyset, X \in \Gamma$;
2. $G_\alpha \in \Gamma, (\alpha \in I), \bigcup_{\alpha\in I} G_\alpha \in \Gamma$ (the union of any family of members of Γ is a member of Γ);
3. $G_k \in \Gamma, \bigcap_{k=1}^{m} G_k \in \Gamma$ (the intersection of finitely many members of Γ is a member of Γ.}

1. $\emptyset, X \in \Gamma$;
2. $G_j \in \Gamma, X\backslash \bigcup_{j\in I} G_j = \bigcap_{j\in I} X\backslash G_j$ is finite, since for each $j \in I$, $X\backslash G_j$ is finite, so $\bigcup_{j\in I} G_j \in \Gamma$;
3. $X\backslash \bigcap_{j=1}^{m} G_j = \bigcup_{j=1}^{m} X\backslash G_j$ is finite, since $X\backslash G_j$ is finite for each $j, j = 1, \cdots, m$, so $\bigcap_{j=1}^{m} G_j \in \Gamma$.

So, Γ is a topology of X, i.e. (X, Γ) is a topology space.

V.

Proof . For any $x \in A$, let $V(x)$ be such neighbourhood of x that $V(x)$ contains only a countable number of points of A, then $A \subset \bigcup V(x)$, that is $\{V(x)\}_{x \in A}$ is a covering of A. According to Lindelöf covering theorem, A has a countable subcovering of $\{V(x)\}_{x \in A}$, denoted by $\{V_\alpha(x)\}_{\alpha \in I}$, $A \subset \bigcup\limits_{\alpha \in I} V_\alpha(x)$, I is a countable set, and since for each $\alpha \in I, V_\alpha(x)$ includes countable number of points, so $\bigcup\limits_{\alpha \in I} V_\alpha(x)$ also includes countable points. Thus A is countable.

VI.

Proof . X is a topological space, Γ is a topology of X, $G_x \in \Gamma$ denotes any element of Γ which includes the point x. $x \in A' = \{\text{the cluster points of} A\} \Longleftrightarrow \forall G_x, G_x \backslash \{x\} \bigcap A \neq \emptyset \Longleftrightarrow G_x \bigcap A \backslash \{x\} \neq \emptyset \Longleftrightarrow x \in \overline{A \backslash \{x\}}$.