Answers to Exercise 4

I.

Proof . Let A be a subset of the metric space E. Step 1: to show $\{x : d(x, A) = 0\} \subset \bar{A}$. $\forall x \in \{x : d(x, A) = 0\}, d(x, A) = \inf_{y \in A}(x, y) = 0$. If $x \in A$, it is obvious that $d(x, A) = 0$; if $x \notin A$, we assert that x must be a cluster point of A, if not, there exists $\delta > 0$, such that $B_d(x, \delta) \cap A = \emptyset$, then $d(x, A) \geq \delta > 0$, which is contradictory to the given condition, so x must be a cluster of A.

Step 2: to show $\bar{A} \subset \{x : d(x, A) = 0\}$. $x \in \bar{A}, x \in A$, or x is the cluster point of A, if $x \in A, d(x, A) = 0$; if x is the cluster point of A, then for any $\varepsilon > 0$, there exists $y \in A$, s.t. $y \in B_d(x, \varepsilon)$, that is $d(x, y) < \varepsilon$, for any $\varepsilon > 0$, so $d(x, A) = \inf_{y \in A} d(x, y) = 0$, $x \in \{x : d(x, A) = 0\}$.

Thus $\{x : d(x, A) = 0\} = \bar{A}$.

II.

Proof . Let $A = \{E_i\}_{i \in I}$ be a collection of open sets in $\mathbb{R}^n$, $E_i \cap E_j = \emptyset, \forall i, j \in I$, then $A = \bigcup_{i \in I} E_i$, that is $\{E_i\}_{i \in I}$ is the open covering of A. According to Lindelöf covering theorem, there exists subcovering of $\{E_i\}_{i \in I}$, denoted by $\{E_n\}_{n \in N}$, which is countable. Since all $E_i, i \in I$ are disjoint with each other, we know that $\{E_i\}_{i \in I} = \{E_n\}_{n \in N}$, that is $I = \mathbb{N}$.

III.

Proof . Method 1. **AT**. If $f(\bar{A})$ is not included in $\overline{f(A)}$, then there exists $x \in \bar{A}$, s.t. $f(x) \notin \overline{f(A)}$, there exists $\varepsilon > 0$, s.t. $B(f(x), \varepsilon) \subset \overline{f(A)}^-$, since $\overline{f(A)}$ is closed. So $f(x) \notin f(A)$, which means $x \notin A$. On the other hand, if $x \in A'(= \bar{A} \backslash A)$, we have a contradiction, since $f^{-1}(B(f(x)), \varepsilon)$ is an open set, and $f^{-1}(B(f(x), \varepsilon)) \backslash \{x\} \bigcap A = \emptyset$, which means $x \notin A'$.

Method 2. $\forall y \in f(\bar{A})$, there exists $x \in \bar{A}$, s.t. $y = f(x)$, because f is continuous, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $f(B(x, \delta)) \subset B(y, \varepsilon)$. Since $x \in \bar{A}$, there exists $x' \in B(x, \delta) \bigcap A$ ($x' = x$, or $x' \neq x$), $f(x') \in f(B(x, \delta)) \subset B(y, \varepsilon)$, and $f(x') \in f(A)$, so $B(y, \varepsilon) \bigcap f(A) \neq \emptyset$, from the arbitrariness of ε, we know $y \in \overline{f(A)}$, thus $f(\bar{A}) \subset \overline{f(A)}$.

Example: Choose $f(x) = \frac{1}{x}$, f is continuous, let $A = \mathbb{N}$, then $f(\bar{A}) = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$, but $\overline{f(A)} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots, 0\}$, $f(\bar{A}) \neq \overline{f(A)}$.

IV.

Proof . $f : X \longrightarrow \mathbb{R}$, we show that $\mathbb{Z}(f)^-$ is open. $\forall x \in \mathbb{Z}(f)^-, f(x) \neq 0$, take $\varepsilon = \frac{1}{2}|f(x)| > 0$, then for any $y \in B(f(x), \varepsilon) = (f(x) - \varepsilon, f(x) + \varepsilon), y \neq 0$, from the continuity of f, there exists $\delta > 0$, s.t. $f(B(x, \delta)) \subset B(f(x), \varepsilon)$, then for any $x' \in B(x, \delta), f(x') \neq 0$, that is $x' \in \mathbb{Z}(f)^-$, which means $B(x, \delta) \subset \mathbb{Z}(f)^-$, thus $\mathbb{Z}(f)^-$ is open, $\mathbb{Z}(f)$ is closed.

Actually, we can apply theorem 1.17 (b)(ii), and get the conclusion immediately, since single point set : $\{0\}$ is a closed set.

V.

Proof . For any open intervals $\{J_n\}$ which such that $B \subset \bigcup_n J_n$, $A \subset \bigcup_n J_n$, since

$A \subset B$. From the definition of outer measure,

$$m^*(A) = \inf\Big\{\sum_n l(I_n) |\ I_n \text{ is open intervals such that } A \subset \bigcup_n I_n\Big\},$$

we know that

$$m^*(A) \leq \sum_n l(J_n),$$

which is hold for any open covering of B, so $m^*(A) \leq \inf\{\sum_n l(J_n)\} = m^*(B)$.

VI.

Proof . $\Big\{$ Corollary 2.4: $A \subset \mathbb{R}$, A is countable, then $m^*(A) = 0\Big\}$. $A = \{a_n\}_{n\in\mathbb{N}}$, let $I_n = (a_n - \frac{\varepsilon}{2^n}, a_n + \frac{\varepsilon}{2^n})$, it is obvious that $\{I_n\}_{n\in\mathbb{N}}$ be the covering of A, $A \subset \bigcup_n I_n$, according to the definition of outer measure, $m^*(A) \leq \sum_n l(I_n) = \sum_{n=1}^{\infty} \frac{2\varepsilon}{2^n} = 2\varepsilon,\ \forall\varepsilon > 0$. So $m^*(A) = 0$.